

# Notes on matrices and calculus

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## 1 Matrices and linear transformations

As usual,  $\mathbf{R}$  and  $\mathbf{C}$  denote the real and complex numbers, respectively. If  $z = x + iy$  is a complex number, with  $x, y \in \mathbf{R}$ , then the *complex conjugate* of  $z$  is denoted  $\bar{z}$  and defined by

$$(1.1) \quad \bar{z} = x - iy.$$

Notice that

$$(1.2) \quad \overline{z + w} = \bar{z} + \bar{w}$$

and

$$(1.3) \quad \overline{\bar{z} \bar{w}} = z w$$

for complex numbers  $z, w$ .

If  $m, n$  are positive integers, we shall denote by  $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$  the space of real-linear mappings from  $\mathbf{R}^m$  to  $\mathbf{R}^n$ , and by  $\mathcal{L}(\mathbf{C}^m, \mathbf{C}^n)$  the space of complex-linear mappings from  $\mathbf{C}^m$  to  $\mathbf{C}^n$ . In the special case where  $m = n$ , we may

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simply write  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$ , respectively. Also when  $m = n$ , we write  $I$  for the identity mapping on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate.

Using the standard basis for real and complex Euclidean spaces, linear transformations can be identified with matrices in the usual manner. Let us write  $\mathbf{M}_r(m, n)$  and  $\mathbf{M}_c(m, n)$  for the spaces of  $m \times n$  real and complex matrices, respectively. Thus  $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^n)$ ,  $\mathcal{L}_c(\mathbf{C}^m, \mathbf{C}^n)$  can be identified with  $\mathbf{M}_r(m, n)$ ,  $\mathbf{M}_c(m, n)$ , respectively, and in particular addition and scalar multiplication of linear transformations corresponds to componentwise addition and scalar multiplication of matrices.

When  $m = n$  we write  $\mathbf{M}_r(n)$  and  $\mathbf{M}_c(n)$  for the spaces of  $n \times n$  real and complex matrices, respectively. Two elements of  $\mathbf{M}_r(n)$  or of  $\mathbf{M}_c(n)$  can be multiplied in the customary manner of “matrix multiplication”, which corresponds exactly to composition of the associated linear transformations on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . The matrix associated to the identity transformation  $I$  has 1’s along the diagonal and 0’s elsewhere, and the product of this matrix with another matrix gives back that other matrix, just as the composition of the identity transformation with another transformation gives back that other transformation.

If  $x = (x_1, \dots, x_n)$ ,  $y = (y_1, \dots, y_n)$  are elements of  $\mathbf{R}^n$ , then their *inner product* is denoted  $\langle x, y \rangle$  and is defined by

$$(1.4) \quad \langle x, y \rangle = \sum_{j=1}^n x_j y_j.$$

In the complex case, the inner product of two vectors  $z = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$  is also denoted  $\langle z, w \rangle$  and is defined by

$$(1.5) \quad \langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}.$$

In both cases the standard Euclidean norm of an element  $v$  of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is denoted  $|v|$  and is defined to be the nonnegative real number such that

$$(1.6) \quad |v|^2 = \langle v, v \rangle.$$

Given a linear transformation  $T$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , there is a unique linear transformation  $T^*$  on the same space such that

$$(1.7) \quad \langle T^*(v), w \rangle = \langle v, T(w) \rangle$$

for all  $v, w$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate. This linear transformation  $T^*$  is called the *adjoint* of  $T$ . In the real case, the matrix associated to  $T^*$  is the *transpose* of the matrix associated to  $T$ , which is to say that the  $(j, l)$  component of the matrix associated to  $T^*$  is equal to the  $(l, j)$  component of the matrix associated to  $T$ ,  $1 \leq j, l \leq n$ , and in the complex case the matrix associated to  $T^*$  can be obtained by taking the complex conjugates of the entries of the transpose of the matrix associated to  $T$ .

A linear transformation  $T$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *self-adjoint* if  $T = T^*$ , and the space of self-adjoint linear transformations on  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  is denoted  $\mathcal{S}(\mathbf{R}^n)$ ,  $\mathcal{S}(\mathbf{C}^n)$ , respectively. The identity transformation is self-adjoint, and if  $T_1, T_2$  are elements of  $\mathcal{S}(\mathbf{R}^n)$  or of  $\mathcal{S}(\mathbf{C}^n)$ , and if  $r_1, r_2$  are real numbers, then the linear combination

$$(1.8) \quad r_1 T_1 + r_2 T_2$$

is also an element of  $\mathcal{S}(\mathbf{R}^n)$  or  $\mathcal{S}(\mathbf{C}^n)$ , respectively. Note that it is important to use real numbers as scalars here even if one is working with linear transformations on  $\mathbf{C}^n$ .

Let us write  $\mathcal{S}_r(n)$ ,  $\mathcal{S}_c(n)$  for the spaces of real and complex  $n \times n$  matrices, respectively, that correspond to self-adjoint linear transformations. Thus  $\mathcal{S}_r(n)$  consists of the matrices in  $\mathbf{M}_r(n)$  which are *symmetric*, in the sense that the  $(j, l)$  and  $(l, j)$  entries are equal to each other. Similarly,  $\mathcal{S}_c(n)$  consists of the matrices in  $\mathbf{M}_c(n)$  such that the  $(l, j)$  entry is equal to the complex conjugate of the  $(j, l)$  entry, and in particular so that the diagonal or  $(j, j)$  entries are real numbers.

If  $x, y \in \mathbf{R}^n$ , then

$$(1.9) \quad \langle y, x \rangle = \langle x, y \rangle,$$

while if  $z, w \in \mathbf{C}^n$ , then

$$(1.10) \quad \langle w, z \rangle = \overline{\langle z, w \rangle}.$$

As a consequence, if  $T$  is a self-adjoint linear transformation on  $\mathbf{C}^n$ , and  $v$  is an element of  $\mathbf{C}^n$ , then

$$(1.11) \quad \overline{\langle T(v), v \rangle} = \langle v, T(v) \rangle = \langle T(v), v \rangle,$$

so that  $\langle T(v), v \rangle$  is a real number. Conversely, if  $T$  is a linear transformation on  $\mathbf{C}^n$  and  $\langle T(v), v \rangle$  is a real number for all  $v \in \mathbf{C}^n$ , then  $T$  is self-adjoint.

In fact, in the complex case a linear transformation  $T$  on  $\mathbf{C}^n$  can always be expressed as  $S_1 + i S_2$ , where  $S_1, S_2$  are self-adjoint linear transformations on  $\mathbf{C}^n$ . Namely, one can take  $S_1 = (T + T^*)/2$  and  $S_2 = (T - T^*)/(2i)$ . It is easy to see that  $\langle T(v), v \rangle$  is real for all  $v \in \mathbf{C}^n$  if and only if  $\langle S_2(v), v \rangle = 0$  for all  $v \in \mathbf{C}^n$ .

In both the real and complex cases we have the following fact. Suppose that  $S$  is a self-adjoint linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  such that

$$(1.12) \quad \langle S(v), v \rangle = 0$$

for all  $v$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate. Then  $S$  is equal to the zero linear transformation.

More generally, suppose that  $S$  is a self-adjoint linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , and that  $v$  is an element of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  such that  $|v| = 1$  and

$$(1.13) \quad \langle S(w), w \rangle$$

is maximized, or minimized, or has a critical point at  $v$ , as a function on the unit sphere, which consists of the vectors  $w$  such that  $|w| = 1$ . As in vector

calculus, one can check that  $v$  is an *eigenvector* for  $S$ , in the sense that there is a scalar  $\lambda$  such that

$$(1.14) \quad S(v) = \lambda v.$$

This scalar  $\lambda$  is called the *eigenvalue* of  $S$  associated to the eigenvector  $v$ , and for a self-adjoint linear transformation it is easy to verify that the eigenvalues must be real numbers, even in the complex case.

This is the computation used in a standard proof of the fact that self-adjoint linear operators on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  can be diagonalized in an orthonormal basis. In other words, if  $S$  is a self-adjoint linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then there are eigenvectors  $v_1, \dots, v_n$  for  $S$  which are orthonormal in the sense that

$$(1.15) \quad \langle v_j, v_l \rangle = 0$$

when  $j \neq l$  and

$$(1.16) \quad \langle v_j, v_j \rangle = 1$$

for each  $j$ . Let us also mention that if  $S$  is a self-adjoint linear transformation on  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  and  $v$  is an eigenvector for  $S$ , and if  $w$  is another vector in  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  which is orthogonal to  $v$  in the sense that

$$(1.17) \quad \langle v, w \rangle = 0,$$

then  $S(w)$  is also orthogonal to  $v$ .

A self-adjoint linear transformation  $T$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *nonnegative* if

$$(1.18) \quad \langle T(v), v \rangle \geq 0$$

for all  $v$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate. This is equivalent to the condition that the eigenvalues of  $T$  be nonnegative real numbers. If  $T_1, T_2$  are nonnegative self-adjoint linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and  $r_1, r_2$  are nonnegative real numbers, then

$$(1.19) \quad r_1 T_1 + r_2 T_2$$

is also a nonnegative self-adjoint linear transformation.

A linear transformation  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *invertible* if there is another linear transformation  $B$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, such that

$$(1.20) \quad A \circ B = B \circ A = I.$$

It is easy to check that if  $B$  is a mapping on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  which is the inverse of  $A$  as a mapping, then  $B$  must also be linear, so that  $A$  is invertible as a linear mapping. The inverse of a linear transformation  $A$  is unique when it exists, and is denoted  $A^{-1}$ .

The *kernel* of a linear transformation  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is the set of vectors  $v$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, such that  $A(v) = 0$ . The kernel of a linear transformation is automatically a linear subspace, which means that it contains the vector 0, the sum of two elements of the kernel again lies in the kernel, and any scalar multiple of a vector in the kernel is also an element of the kernel.

The kernel of a linear transformation is said to be *trivial* if it contains only the vector 0.

If a linear transformation is invertible, then its kernel is trivial. Conversely, if  $A$  is a linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  whose kernel is trivial, then  $A$  is invertible. This is a well known fact from linear algebra, and similarly  $A$  is invertible if and only if it maps  $\mathbf{R}^n$  or  $\mathbf{C}^n$  onto itself, as appropriate.

The statement that a linear transformation  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is nontrivial is equivalent to the statement that  $A$  has a nonzero eigenvector with eigenvalue equal to 0. More generally, a scalar  $\lambda$  is an eigenvalue for a linear transformation  $A$  if and only if the linear transformation

$$(1.21) \quad A - \lambda I$$

has a nontrivial kernel. For the record, a scalar  $\lambda$  is considered to be an eigenvalue of a linear transformation  $A$  only when there is a *nonzero* eigenvector for  $A$  with eigenvalue  $\lambda$ .

If  $A_1, A_2$  are invertible linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then the composition  $A_1 \circ A_2$  is also invertible. In this case we have that

$$(1.22) \quad (A_1 \circ A_2)^{-1} = A_2^{-1} \circ A_1^{-1}.$$

Conversely, if  $A_1$  and  $A_2$  are linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  such that  $A_1 \circ A_2$  is invertible, then  $A_1$  and  $A_2$  are each invertible themselves, because  $A_1$  maps  $\mathbf{R}^n$  or  $\mathbf{C}^n$  onto itself, as appropriate, and  $A_2$  has trivial kernel.

Suppose that  $T_1, T_2$  are linear operators on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ . One can check that

$$(1.23) \quad (T_1 \circ T_2)^* = T_2^* \circ T_1^*.$$

If  $T$  is an invertible linear operator on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then  $T^*$  is also invertible, with

$$(1.24) \quad (T^*)^{-1} = (T^{-1})^*,$$

and in particular the inverse of an invertible self-adjoint linear operator is also self-adjoint.

A self-adjoint linear operator  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *positive-definite* if

$$(1.25) \quad \langle A(v), v \rangle > 0$$

for all nonzero vectors  $v$ . Thus a positive-definite self-adjoint linear operator is invertible, because it has trivial kernel, and one can check that the inverse is also positive-definite. Also, a self-adjoint linear transformation is positive-definite if and only if it is nonnegative and invertible.

Suppose that  $T$  is any linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . Clearly  $T^* \circ T$  is self-adjoint, and it is nonnegative as well. Moreover,  $T^* \circ T$  is positive definite if and only if  $T$  is invertible.

If  $A$  is a self-adjoint linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  which is positive-definite, and if  $\alpha$  is a positive real number, then  $\alpha A$  is also a self-adjoint linear transformation which is positive-definite. If  $A_1, A_2$  are two self-adjoint linear

transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  which are self-adjoint and nonnegative, and if at least one of  $A_1, A_2$  is positive-definite, then the sum  $A_1 + A_2$  is a self-adjoint linear transformation which is positive-definite. In particular, the sum of two self-adjoint linear transformations which are positive-definite is again positive-definite.

A linear transformation  $T$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *orthogonal* or *unitary*, respectively, if  $T$  is invertible and

$$(1.26) \quad T^{-1} = T^*.$$

This is equivalent to saying that

$$(1.27) \quad \langle T(v), T(w) \rangle = \langle v, w \rangle$$

for all vectors  $v, w$  in the domain. In fact, this is equivalent to

$$(1.28) \quad |T(v)| = |v|$$

for all vectors  $v$  in the domain, as one can show using *polarization*.

A linear transformation  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *anti-self-adjoint* if

$$(1.29) \quad A^* = -A.$$

Any linear transformation  $T$  can be written as  $S + A$ , with  $S$  a self-adjoint linear transformation and  $A$  an anti-self-adjoint linear transformation, simply by taking

$$(1.30) \quad S = \frac{T + T^*}{2}, \quad A = \frac{T - T^*}{2}.$$

In the complex case a linear transformation is anti-self-adjoint if and only if it is  $i$  times a self-adjoint linear transformation, and in both the real and complex cases it can be useful to observe that the square of an anti-self-adjoint operator is self-adjoint, and in fact it is  $-1$  times a nonnegative self-adjoint operator.

As above, a subset  $L$  of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be a *linear subspace* if  $0 \in L$ ,  $v, w \in L$  implies  $v + w \in L$ , and  $v \in L$  implies  $\alpha v \in L$  for all scalars  $\alpha$ , which is to say all real or complex numbers, as appropriate. The subspace consisting of only the vector  $0$  is called the trivial subspace. Of course  $\mathbf{R}^n, \mathbf{C}^n$  are linear subspaces of themselves.

Suppose that  $v_1, \dots, v_m$  is a finite collection of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . The *span* of  $v_1, \dots, v_m$  is denoted

$$(1.31) \quad \text{span}\{v_1, \dots, v_m\}$$

and is the linear subspace consisting of the vectors of the form

$$(1.32) \quad \sum_{j=1}^m \alpha_j v_j,$$

where  $\alpha_1, \dots, \alpha_m$  are scalars. A linear subspace  $L$  of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be spanned by a finite collection of vectors  $v_1, \dots, v_m \in L$  if the span of those vectors is equal to  $L$ .

A finite collection  $v_1, \dots, v_m$  of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *linearly independent* if a linear combination  $\sum_{j=1}^m \alpha_j v_j$  of the  $v_j$ 's is equal to the vector 0 only when the scalars  $\alpha_j$  are all equal to 0. This is equivalent to saying that vectors in the span of  $v_1, \dots, v_m$  are represented in a unique manner as a linear combination  $\sum_{j=1}^m \alpha_j v_j$ . A finite collection  $\{v_1, \dots, v_m\}$  of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be a *basis* for a linear subspace  $L$  if  $v_1, \dots, v_m$  are linearly independent and their span is equal to  $L$ .

A finite collection  $v_1, \dots, v_m$  of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is linearly dependent if there are scalars  $\alpha_1, \dots, \alpha_m$ , with  $\alpha_j \neq 0$  for at least one  $j$ , such that

$$(1.33) \quad \sum_{j=1}^m \alpha_j v_j = 0.$$

In this case one can reduce the collection to a smaller one with the same span, at least if we consider the trivial subspace to be the span of the empty collection of vectors. Assuming that at least one of the vectors is nonzero, we can repeat the process to obtain a nonempty subcollection of vectors which is linearly independent and has the same span.

A basic result from linear algebra states that if  $L$  is a linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  which is spanned by a collection of  $m$  vectors, then every linearly independent collection of vectors in  $L$  has less than or equal to  $m$  elements. This comes down to the fact that a system of  $l$  homogeneous linear equations with more than  $l$  variables always has a nontrivial solution. One can turn this around and say that if  $L$  contains a set of  $k$  linearly independent vectors, then any collection of vectors which spans  $L$  has at least  $k$  elements.

The *standard basis* for  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is the collection of  $n$  vectors, each of which has exactly one component equal to 1 and the others equal to 0. It is easy to see that this is a basis, which is to say that it is linearly independent and spans the whole space. Also, every linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is spanned by a finite collection of vectors, and hence has a basis, using the empty collection of vectors for the trivial subspace.

The *dimension* of a linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is equal to the number of elements of a basis in the subspace. By the earlier remarks this number is the same for each basis. The dimension can also be described as the maximum number of linearly independent vectors in the subspace, or the minimal number of vectors needed to span the subspace.

Let  $L$  be a linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  with dimension  $l$ . A collection of  $l$  linearly independent vectors in  $L$  also spans  $L$ , since otherwise one could add a vector in  $L$  not in the span of these vectors to get a collection of  $l + 1$  linearly independent vectors in  $L$ . Similarly, a collection of  $l$  vectors in  $L$  which spans  $L$  is also linearly independent.

Suppose that  $T$  is a linear operator on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , and that  $L$  is a linear subspace of the same space. In this event  $T(L)$ , the image of  $L$  under  $T$ , is also a linear subspace. If  $T$  is invertible, then the dimension of  $T(L)$  is equal to the dimension of  $L$ .

A collection of vectors  $v_1, \dots, v_m$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is said to be *orthonormal* if, as before,

$$(1.34) \quad \langle v_j, v_k \rangle = 0$$

when  $j \neq k$  and

$$(1.35) \quad |v_j| = 1$$

for each  $j$ . If  $v_1, \dots, v_m$  is an orthonormal collection of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  and  $w$  is in their span, so that

$$(1.36) \quad w = \sum_{j=1}^m \alpha_j v_j$$

for some scalars  $\alpha_j$ , then

$$(1.37) \quad \alpha_j = \langle w, v_j \rangle$$

for each  $j$ , and in particular  $v_1, \dots, v_m$  are linearly independent. Also, we have that

$$(1.38) \quad |w|^2 = \sum_{j=1}^m |\langle w, v_j \rangle|^2$$

in this case.

Let us recall the *Cauchy-Schwarz inequality*, which states that if  $v, w$  are elements of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$ , then

$$(1.39) \quad |\langle v, w \rangle| \leq |v| |w|.$$

This can be shown using the fact that

$$(1.40) \quad \langle v + \alpha w, v + \alpha w \rangle = |v + \alpha w|^2 \geq 0$$

for all scalars  $\alpha$ . Using this inequality, one can also show that

$$(1.41) \quad |v + w| \leq |v| + |w|,$$

which is to say the triangle inequality.

As before, if  $v, w$  are two vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then we say that  $v, w$  are *orthogonal* if

$$(1.42) \quad \langle v, w \rangle = 0,$$

and in this case we write  $v \perp w$ . If  $v, w$  are orthogonal vectors in  $\mathbf{R}^n$  or in  $\mathbf{C}^n$ , then

$$(1.43) \quad |v + w|^2 = |v|^2 + |w|^2,$$

and  $\alpha v \perp \beta w$  for all scalars  $\alpha, \beta$ . Conversely, notice that if  $v, w$  are two vectors in  $\mathbf{R}^n$  such that  $|v + w|^2 = |v|^2 + |w|^2$ , then  $v \perp w$ , and if  $v, w$  are two vectors in  $\mathbf{C}^n$  such that  $|v + \alpha w|^2 = |v|^2 + |w|^2$  for all complex numbers  $\alpha$  with  $|\alpha| = 1$ , then  $v \perp w$ .



Suppose again that  $v_1, \dots, v_m$  is an orthonormal collection of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . If  $u$  is any vector in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then

$$(1.44) \quad u' = \sum_{j=1}^m \langle u, v_j \rangle v_j$$

lies in the span of  $v_1, \dots, v_m$ , and one can check that  $u - u'$  is orthogonal to every vector in the linear span of  $v_1, \dots, v_m$ . In particular,

$$(1.45) \quad \langle u - u', u' \rangle = 0,$$

and

$$(1.46) \quad |u|^2 = |u - u'|^2 + |u'|^2.$$

If  $T$  is a linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then the *trace* of  $T$  is denoted  $\text{tr } T$  and is defined to be the sum of the diagonal terms in the standard matrix associated to  $T$ . To be more explicit, let  $e_1, \dots, e_n$  denote the standard basis for  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, so that  $e_j$  has  $j$ th component equal to 1 and all other components equal to 0. The trace of a linear transformation  $T$  can then be expressed as

$$(1.47) \quad \text{tr } T = \sum_{j=1}^n \langle T(e_j), e_j \rangle.$$

The trace is clearly linear in  $T$ , so that if  $T_1, T_2$  are linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and  $\alpha_1, \alpha_2$  are scalars, then

$$(1.48) \quad \text{tr}(\alpha_1 T_1 + \alpha_2 T_2) = \alpha_1 \text{tr } T_1 + \alpha_2 \text{tr } T_2.$$

Another fundamental property of the trace is that

$$(1.49) \quad \text{tr}(T_1 \circ T_2) = \text{tr}(T_2 \circ T_1)$$

for all linear transformations  $T_1, T_2$ . This can be verified in a straightforward manner.

If  $T$  is a linear transformation on  $\mathbf{R}^n$ , then

$$(1.50) \quad \text{tr } T^* = \text{tr } T,$$

while if  $T$  is a linear transformation on  $\mathbf{C}^n$ , then

$$(1.51) \quad \text{tr } T^* = \overline{\text{tr } T}.$$

If  $A, B$  are linear transformations on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then

$$(1.52) \quad \text{tr}(B^* \circ A) = \sum_{j=1}^n \langle B^*(A(e_j)), e_j \rangle = \sum_{j=1}^n \langle A(e_j), B(e_j) \rangle.$$

where as usual  $e_1, \dots, e_n$  denotes the standard basis for  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . This is the same as

$$(1.53) \quad \sum_{j=1}^n \sum_{l=1}^n \langle A(e_j), e_l \rangle \langle B(e_j), e_l \rangle$$

in the real case and

$$(1.54) \quad = \sum_{j=1}^n \sum_{l=1}^n \langle A(e_j), e_l \rangle \overline{\langle B(e_j), e_l \rangle}$$

in the complex case, which is to say that one takes the standard matrices of  $A$ ,  $B$ , views them as elements of  $\mathbf{R}^{n^2}$  or  $\mathbf{C}^{n^2}$ , as appropriate, and then takes the usual inner product.

In particular, if  $T$  is a linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , let  $\|T\|_{HS}$  be the nonnegative real number defined by

$$(1.55) \quad \|T\|_{HS}^2 = \text{tr}(T^* \circ T) = \sum_{j=1}^n \sum_{k=1}^n |\langle T(e_j), e_k \rangle|^2.$$

In other words,  $\|T\|_{HS}$  is the same as the usual Euclidean norm of the standard matrix associated to  $T$ , and it is also known as the *Hilbert–Schmit norm* of  $T$ . Observe that  $\|T\|_{HS} = 0$  if and only if  $T = 0$ ,  $\|\alpha T\|_{HS} = |\alpha| \|T\|_{HS}$  for all scalars  $\alpha$  and all linear transformations  $T$ ,  $\|T_1 + T_2\|_{HS} \leq \|T_1\|_{HS} + \|T_2\|_{HS}$  for all linear transformations  $T_1, T_2$ , and that

$$(1.56) \quad |\text{tr}(T_2^* \circ T_1)| \leq \|T_1\|_{HS} \|T_2\|_{HS}$$

for all linear transformations  $T_1, T_2$ .

If  $T$  is a linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then the *operator norm* of  $T$  is denoted  $\|T\|_{op}$  and defined to be the maximum of  $|T(v)|$  over all vectors  $v$  in the domain with  $|v| = 1$ , which exists by the extreme value theorem in calculus. In other words,

$$(1.57) \quad |T(w)| \leq \|T\|_{op} |w|$$

for all vectors  $w$ , and  $\|T\|_{op}$  is the smallest nonnegative real number with this property. One can check that  $\|T\|_{op} = 0$  if and only if  $T = 0$ ,  $\|\alpha T\|_{op} = |\alpha| \|T\|_{op}$  for all scalars  $\alpha$  and all linear transformations  $T$ ,  $\|T_1 + T_2\|_{op} \leq \|T_1\|_{op} + \|T_2\|_{op}$  for all linear transformations  $T_1, T_2$ , and

$$(1.58) \quad \|T_1 \circ T_2\|_{op} \leq \|T_1\|_{op} \|T_2\|_{op}$$

for all linear transformations  $T_1, T_2$ .

Alternatively,  $\|T\|_{op}$  can be described as the maximum of  $|\langle T(v), w \rangle|$  over all vectors  $v, w$  in the domain such that  $|v| = |w| = 1$ , which is the same as saying that

$$(1.59) \quad |\langle T(v), w \rangle| \leq \|T\|_{op} |v| |w|$$

for all vectors  $v, w$  in the domain, and that  $\|T\|_{op}$  is the smallest nonnegative real number with this property. In particular, it follows that

$$(1.60) \quad \|T^*\|_{op} = \|T\|_{op}$$

for all linear transformations  $T$ . It is easy to check as well that

$$(1.61) \quad \|T^*\|_{HS} = \|T\|_{HS}$$

for all linear transformations  $T$ .

Notice that

$$(1.62) \quad |\langle T(e_j), e_k \rangle| \leq \|T\|_{op},$$

so that the operator norm of  $T$  is greater than or equal to the absolute values of the entries of the standard matrix associated to  $T$ . One can express  $\|T\|_{HS}$  by

$$(1.63) \quad \|T\|_{HS}^2 = \sum_{j=1}^n |T(e_j)|^2,$$

from which it follows that  $\|T\|_{HS} \leq n^{1/2} \|T\|_{op}$ . From this formula it also follows that  $\|A \circ B\|_{HS} \leq \|A\|_{op} \|B\|_{HS}$ , and similarly one has  $\|A \circ B\|_{HS} \leq \|A\|_{HS} \|B\|_{op}$  for all linear transformations  $A, B$ .

Suppose that  $v_1, \dots, v_m$  is an orthonormal collection of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , and let  $L$  denote the span of this collection. As we have seen, if  $u$  is any vector in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, then there is a vector  $u' \in L$  such that  $u - u'$  is orthogonal to every element of  $L$ . These two properties characterize  $u'$ , since if  $u'' \in L$  and  $u - u''$  is orthogonal to every element of  $L$ , then  $u' - u''$  is an element of  $L$  and is orthogonal to every element of  $L$ , including itself, so that  $u' - u'' = 0$ .

In this situation let us write  $P_L$  for the linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, which sends  $u$  to  $u'$ . This is called the *orthogonal projection* of  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, onto  $L$ . It is uniquely determined by  $L$ , which is to say that it does not depend on the choice of orthonormal basis for  $L$ .

Using these orthogonal projections, one can show that every orthonormal set of vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  can be extended to an orthonormal basis, and that every linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  has an orthonormal basis. This is basically the same as the Gram–Schmit process, in which a collection of vectors is orthonormalized one step at a time. In particular, for every linear subspace  $L$  of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  there is a corresponding orthogonal projection  $P_L$ , which one can also check is self-adjoint.

Let  $v_1, \dots, v_n$  and  $w_1, \dots, w_n$  be orthonormal bases of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$ , respectively. If  $T$  is a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , as appropriate, then one can check that

$$(1.64) \quad \|T\|_{HS}^2 = \sum_{j=1}^n |T(v_j)|^2$$

and that

$$(1.65) \quad \|T\|_{HS}^2 = \sum_{j=1}^n \sum_{k=1}^n |\langle T(v_j), w_k \rangle|^2.$$

In particular, it follows that  $\|T\|_{op} \leq \|T\|_{HS}$ .

Suppose that  $L$  is a nontrivial linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , and that  $P_L$  is the corresponding orthogonal projection onto  $L$ . For each vector  $u$  in the domain, we have that  $P_L(u)$  and  $u - P_L(u)$  are orthogonal to each other, so that

$$(1.66) \quad |P_L(u)|^2 + |u - P_L(u)|^2 = |u|^2,$$

and one can check that  $\|P_L\|_{op} = 1$ . From the remarks in the previous paragraphs it follows that  $\|P_L\|_{HS}$  is equal to the square root of the dimension of  $L$ .

In general, a *projection* on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  is a linear operator  $P$  which is an “idempotent”, which means that

$$(1.67) \quad P^2 = P.$$

Thus for instance the identity and the operator 0 are projections, and in general if  $P$  is a projection and  $L$  is the image of  $P$ , so that  $L$  consists of the vectors of the form  $P(v)$  for vectors  $v$  in the domain, then  $L$  is exactly the set of vectors  $w$  such that  $P(w) = w$ . If  $P$  is a projection and  $v$  is any vector in the domain, then  $P(v)$  lies in the image of  $P$  and  $v - P(v)$  lies in the kernel of  $P$ .

If  $L$  is a linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then the *orthogonal complement* of  $L$  is denoted  $L^\perp$  and defined to be the linear subspace of vectors  $v$  such that  $v$  is orthogonal to  $w$  for all  $w \in L$ . From the earlier remarks it follows that every vector  $u$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, can be written in a unique way as the sum of vectors in  $L$  and  $L^\perp$ . One can also check that  $(L^\perp)^\perp = L$ .

A projection  $P$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  with image  $L$  is equal to the orthogonal projection onto  $L$  if and only if the kernel of  $P$  is equal to  $L^\perp$ . Also, a projection  $P$  is an orthogonal projection if and only if  $P$  is self-adjoint. The operator norm of a nonzero projection is automatically greater than or equal to 1, and one can check that it is equal to 1 if and only if the projection is an orthogonal projection.

Now let us briefly review some aspects of *determinants*. We begin with some facts about *permutations*. Fix a positive integer  $n$ , and let  $\text{Sym}(n)$  denote the *symmetric group on*  $\{1, \dots, n\}$  consisting of the permutations on the set  $\{1, \dots, n\}$  of positive integers from 1 to  $n$ , which is to say the one-to-one mappings from this set onto itself, with composition mappings as the group operation, and inverses of mappings as inverses in the group.

A *transposition* is a permutation  $\tau$  on  $\{1, \dots, n\}$  which interchanges two elements of the set and leaves the others fixed. A basic fact is that every element of the symmetric group can be expressed as a composition of finitely many transpositions. Of course such a product is not unique, and another important result is that the parity of the number of transpositions used is unique, i.e., it depends only on the original permutation.

In effect this is the same as saying that the identity permutation, which fixes all elements of the set, can be expressed as a composition of an even number of transpositions, and not an odd number of transpositions. An element of the symmetric group is said to be even or odd according to whether it can be expressed as the composition of an even or odd number of transpositions. The composition of two even permutations is even, the composition of two odd permutations is an odd permutation, the composition of an even and an odd permutation is an odd permutation, and the inverse of a permutation  $\pi$  has the same type as  $\pi$  does.

Now let  $A$  be a linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , and let  $(a_{j,l})$  denote the corresponding  $n \times n$  matrix of real or complex numbers. The determinant of  $A$

is denoted  $\det A$  and is the real or complex number, respectively, given by

$$(1.68) \quad \det A = \sum_{\pi \in \text{Sym}(n)} \text{sign}(\pi) a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)},$$

where  $\text{sign}(\pi)$  is equal to  $+1$  or  $-1$  according to whether the permutation  $\pi$  is even or odd. Thus the determinant of  $A$  is a homogeneous polynomial of degree  $n$  as a function of the entries of the matrix  $(a_{j,l})$ .

When  $n = 1$ , the matrix associated to  $A$  is really just a single number, and the determinant of  $A$  is that number. In general we have that  $\det I = 1$ ,  $\det A^* = \det A$  for all  $A$ , and

$$(1.69) \quad \det(A \circ B) = (\det A)(\det B)$$

for all linear transformations  $A, B$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ . It follows from this that if  $A$  is an invertible linear transformation, then  $\det A \neq 0$  and indeed

$$(1.70) \quad (\det A)^{-1} = \det(A^{-1}),$$

and conversely there is the well-known *Cramer's rule*, which states that a linear transformation with nonzero determinant is invertible, with a formula for the inverse in terms of the determinant of the linear transformation and determinants of submatrices of the associated matrix.

Let  $v_1, \dots, v_n$  be a basis for  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , and let  $A$  be a linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . It is easy to see that  $A$  is uniquely determined by its values on  $v_1, \dots, v_n$ , and conversely that if  $w_1, \dots, w_n$  is any other collection of  $n$  vectors in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, then there is a linear transformation  $A$  such that  $A(v_j) = w_j$  for each  $j$ . Also,  $A$  is an invertible linear transformation if and only if  $A(v_1), \dots, A(v_n)$  is a basis too.

For any choice of basis for  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , there is a natural correspondence between linear transformations on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  and matrices with real or complex entries, respectively, in such a way that the diagonal matrices correspond exactly to linear transformations for which the vectors in the basis are eigenvectors. Of course for any two choices of bases there is an invertible linear transformation which sends one basis to the other. For a single linear transformation, one gets two matrices associated to the two bases, and these two matrices are related by conjugation.

In particular, for a linear transformation  $A$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and a choice of basis  $v_1, \dots, v_n$ , one gets a matrix associated to this linear transformation and basis, and one can take the trace or determinant of this matrix. It follows from the basic identities for the trace and determinant that the trace of this matrix is the same as for the matrix associated to any other choice of basis. As a special case, if the linear transformation is diagonalizable, in the sense that there is a basis of eigenvectors, then the trace is the same as the sum of the corresponding  $n$  eigenvalues, and the determinant is equal to the product of the eigenvalues.

As another basic example, if  $P$  is a projection on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  whose image is a linear subspace  $L$  of dimension  $l$ , then the trace of  $P$  is equal to  $l$ .

Now let us look at exponentials, beginning with exponentiation of real numbers. The exponential function is denoted  $\exp(x)$  and can be defined by the series expansion

$$(1.71) \quad \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Here  $x^n$  is interpreted as being equal to 1 when  $n = 0$ , even if  $x = 0$ , and  $n!$  is “ $n$  factorial”, the product of the positive integers from 1 to  $n$ , which is also interpreted as being equal to 1 when  $n = 0$ .

By standard results, this series converges for all  $x \in \mathbf{R}$ , and converges absolutely, and it also converges uniformly on bounded subsets of  $\mathbf{R}$ . The sum defines a real-valued function on the real line which is continuous and has continuous derivatives of all orders, with the derivatives being given by the series obtained by differentiating this one term by term. In this case we have the well-known identity

$$(1.72) \quad \exp'(x) = \exp(x),$$

i.e., the derivative of the exponential function is itself.

A related identity is

$$(1.73) \quad \exp(x + y) = \exp(x) \exp(y).$$

Formally this can be derived by multiplying the series for  $\exp(x)$  and  $\exp(y)$ , group terms of total degree  $n$ , and using the binomial theorem to identify them with the terms of  $\exp(x + y)$ . Convergence issues can be handled using absolute convergence of the series involved, by standard arguments.

Clearly  $\exp(x) \geq 1$  when  $x \geq 0$ . From the multiplicative identity it follows that  $\exp(x) \neq 0$  for all  $x \in \mathbf{R}$ , and in fact that

$$(1.74) \quad \exp(-x) = \frac{1}{\exp(x)}.$$

It follows that  $\exp(x) > 0$  for all  $x \in \mathbf{R}$ , and hence the derivatives of  $\exp(x)$  are all positive as well, so that  $\exp(x)$  is strictly increasing and strictly convex in particular.

Next we consider complex numbers. That is, we define  $\exp(z)$  for  $z \in \mathbf{C}$  by the same series as before, namely,

$$(1.75) \quad \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

This series converges absolutely for all  $z \in \mathbf{C}$ , it converges uniformly on bounded subsets of  $\mathbf{C}$ , and it is continuously differentiable of all orders.

Again we have the identities

$$(1.76) \quad \exp'(z) = \exp(z)$$

and

$$(1.77) \quad \exp(z + w) = \exp(z) \exp(w)$$

for all  $z, w \in \mathbf{C}$ . The meaning of the differential equation for  $\exp(z)$  is that  $\exp(z)$  is a holomorphic function of  $z$  whose complex derivative is equal to  $\exp(z)$ . To put it another way, the differential of  $\exp(z)$  at a point  $z$  is given by multiplication by  $\exp(z)$ , so that

$$(1.78) \quad \exp(z+h) = \exp(z) + \exp(z) \cdot h + O(h^2).$$

If  $z = x + iy$ , with  $x, y \in \mathbf{R}$ , then

$$(1.79) \quad \exp(z) = \exp(x) (\cos(y) + i \sin(y)).$$

This is a well-known and striking formula, which can be seen by writing out the series expansions for the real and imaginary parts of  $\exp(iy)$  and comparing them with the usual series expansions for the cosine and sine. Also, as a complex-valued function of a real variable, we have that

$$(1.80) \quad \frac{d}{dy} \exp(iy) = i \exp(iy)$$

and hence

$$(1.81) \quad \frac{d^2}{dy^2} \exp(iy) = -\exp(iy),$$

which correspond to standard formulas for the derivatives of the cosine and sine, including the second-order differential equations that they satisfy.

It is clear from the series expansion that

$$(1.82) \quad \overline{\exp(z)} = \exp(\bar{z})$$

for all  $z \in \mathbf{C}$ . In particular, if  $z = x + iy$  with  $x, y \in \mathbf{R}$ , then

$$(1.83) \quad |\exp(z)| = \exp(x).$$

The special case

$$(1.84) \quad |\exp(iy)| = 1$$

for all  $y \in \mathbf{R}$  corresponds to the usual identity  $\cos(y)^2 + \sin(y)^2 = 1$ .

Fix a positive integer  $n$ , and suppose that  $A$  is a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ . We would like to define  $\exp(A)$  by the series

$$(1.85) \quad \exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!},$$

where now  $A^k$  denotes the  $k$ -fold composition of  $A$  as a linear transformation, interpreted as being the identity operator  $I$  when  $k = 0$ . The convergence of this series can be defined in terms of the convergence of the entries of the corresponding matrices, and as before we have absolute convergence for all linear transformations  $A$ , uniform convergence on bounded sets of such linear transformations, and that the exponential defines a continuous function from linear transformations to themselves which is continuously differentiable of all orders.

A convenient way to look at absolute convergence of series of linear transformations is in terms of convergence of the corresponding series of operator norms. In this case we have such convergence, because

$$(1.86) \quad \sum_{k=0}^{\infty} \frac{\|A^k\|_{op}}{k!} \leq \sum_{k=0}^{\infty} \frac{\|A\|_{op}^k}{k!}.$$

In particular, note that

$$(1.87) \quad \|\exp(A)\|_{op} \leq \exp(\|A\|_{op}),$$

where the right side refers to the exponential of the operator norm of  $A$  as a real number.

If  $A$  and  $B$  are linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  which commute, then we still have that

$$(1.88) \quad \exp(A + B) = \exp(A) \circ \exp(B),$$

for essentially the same reasons as before. Of course

$$(1.89) \quad \exp(0) = I,$$

and for any linear transformation  $A$  we have that  $\exp(A)$  is invertible, with

$$(1.90) \quad (\exp(A))^{-1} = \exp(-A)$$

and thus

$$(1.91) \quad \|(\exp(A))^{-1}\|_{op} \leq \exp\|A\|_{op}.$$

Furthermore,

$$(1.92) \quad (\exp(A))^* = \exp(A^*),$$

which is to say that the adjoint of the exponential of  $A$  is equal to the exponential of the adjoint of  $A$ .

If  $A$  is a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , and if  $T$  is another linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , as appropriate, which commutes with  $A$ , then  $T$  also commutes with  $\exp(A)$ , and the directional derivative of  $\exp(A)$  at  $A$  in the direction of  $T$  is given by multiplication by  $\exp(A)$ , so that

$$(1.93) \quad \exp(A + T) = \exp(A) + \exp(A)T + O(\|T\|_{op}^2).$$

In particular, if  $A$  is a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and we put

$$(1.94) \quad E_A(t) = \exp(tA),$$

viewed as a function on the real line with values in linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , as appropriate, then this function is continuously differentiable of all orders and satisfies

$$(1.95) \quad \frac{d}{dt}E_A(t) = A \circ E_A(t), \quad E_A(0) = I.$$



These conditions characterize  $E_A(t)$  uniquely, by standard results about ordinary differential equations.

What about the determinant of the exponential of a linear transformation? Notice first that the differential of the determinant as a function on linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and evaluated at the identity transformation is given by the trace. That is, if  $T$  is any linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then

$$(1.96) \quad \det(I + T) = 1 + \operatorname{tr} T + O(\|T\|_{op}^2),$$

and of course this is just a simple algebraic statement, since the determinant of a linear transformation  $A$  is a polynomial in the entries of the matrix associated to  $A$ .

This implies that

$$(1.97) \quad \frac{d}{dt} \det(\exp(tA)) = (\operatorname{tr} A) \det(\exp(tA)).$$

More precisely, at  $t = 0$  this follows exactly from the remarks of the preceding paragraph. In general, for any two real numbers  $r, s$ , we have that

$$(1.98) \quad \exp((r + s)A) = \exp(rA) \circ \exp(sA),$$

and this permits one to derive the formula for the derivative at any real number  $t$  from the special case of  $t = 0$ .

Of course the determinant of  $\exp(tA)$  at  $t = 0$  is equal to 1, and it follows that

$$(1.99) \quad \det(\exp(tA)) = \exp(t \operatorname{tr} A).$$

The trace of  $A$  is a real or complex number, and the right side is the usual exponential of a scalar. We may as well apply this to  $t = 1$  and say that

$$(1.100) \quad \det(\exp(A)) = \exp(\operatorname{tr} A).$$

Suppose that  $A$  is a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and that  $v$  is a vector in  $\mathbf{R}^n$  or in  $\mathbf{C}^n$ , as appropriate. Set

$$(1.101) \quad h(t) = \exp(tA)(v)$$

viewed as a function from the real line into  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , i.e., where for each  $t$  we let  $h(t)$  be the image of  $v$  under  $\exp(tA)$ . As before we have that  $h'(t) = A(h(t))$  and that  $h(0) = v$ , and  $h(t)$  is characterized by these properties by standard results about ordinary differential equations.

Assume further that  $v$  is an eigenvector for  $A$  with eigenvalue  $\lambda$ , so that

$$(1.102) \quad A(v) = \lambda v,$$

where  $\lambda$  is a scalar. In this case

$$(1.103) \quad \exp(tA)(v) = \exp(t\lambda)v,$$

where  $\exp(t\lambda)$  is the usual exponential mapping for scalars. One can see this either from the series expansion for the exponential or from the characterization in terms of ordinary differential equations.

It may be that  $A$  is diagonalizable, so that there is a basis of eigenvectors for  $A$ . The elements of this basis are then eigenvectors for  $\exp(tA)$  too, with the eigenvalues for the exponential being given by the exponentials of the corresponding eigenvalues, as in the previous paragraph. In other words, the exponential is then also diagonalizable, and by the same basis as for  $A$  itself.

For that matter, suppose that  $A$  is a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , and that  $L$  is a linear subspace of the same space which is invariant under  $A$ . This means that

$$(1.104) \quad A(L) \subseteq L,$$

which is to say that  $A(v) \in L$  for all  $v \in L$ . In this event  $L$  is invariant under  $\exp(tA)$  for all  $t$  as well, as one can see from either the series expansion for the exponential or the characterization in terms of ordinary differential equations.

Next we review some aspects of spectral theory of matrices. If  $A$  is a linear transformation on  $\mathbf{C}^n$ , then the *characteristic polynomial* associated to  $A$  is defined by

$$(1.105) \quad q_A(z) = \det(zI - A).$$

Thus  $q_A(z)$  is a polynomial of degree  $n$  whose leading coefficient is equal to 1 and which vanishes exactly at the eigenvalues of  $A$ .

The fundamental theorem of algebra states that every nonconstant polynomial on the complex numbers has a root. As a result, every linear transformation on  $\mathbf{C}^n$  has at least one eigenvalue. Recall as well that every nonconstant polynomial on the complex numbers can be factored as a nonzero complex number times a product of linear factors of the form  $(z - \alpha)$ ,  $\alpha \in \mathbf{C}$ .

If  $p(z)$  is a polynomial,

$$(1.106) \quad p(z) = c_m z^m + c_{m-1} z^{m-1} + \cdots + c_0,$$

$c_0, \dots, c_m \in \mathbf{C}$ , and  $A$  is a linear transformation on  $\mathbf{C}^n$ , then we can define  $p(A)$  to be the linear transformation on  $\mathbf{C}^n$  given by

$$(1.107) \quad p(A) = c_m A^m + c_{m-1} A^{m-1} + \cdots + c_0 I.$$

Notice that if  $p_1, p_2$  are polynomials, so that the sum  $p_1 + p_2$  and the product  $p_1 p_2$  are also polynomials, then we have that

$$(1.108) \quad (p_1 + p_2)(A) = p_1(A) + p_2(A)$$

and

$$(1.109) \quad (p_1 p_2)(A) = p_1(A) p_2(A) = p_2(A) p_1(A).$$

Moreover, the composition  $p_1 \circ p_2$  is also a polynomial, and  $(p_1 \circ p_2)(A) = p_1(p_2(A))$ .

If  $A$  is a linear transformation on  $\mathbf{C}^n$ ,  $v$  is a vector in  $\mathbf{C}^n$  which is an eigenvector for  $A$  with eigenvalue  $\lambda$ , and  $p(z)$  is a polynomial, then  $v$  is also an

eigenvector for  $p(A)$ , with eigenvalue  $p(\lambda)$ . Conversely, if  $A$  is a linear transformation on  $\mathbf{C}^n$  and  $h(z)$  is a polynomial such that  $h(\lambda) \neq 0$  for all eigenvalues  $\lambda$  of  $A$ , then  $h(A)$  is invertible. As a consequence, for a linear transformation  $A$  on  $\mathbf{C}^n$  and a polynomial  $p(z)$ , if a complex number  $\mu$  is an eigenvalue of  $p(A)$ , then there is an eigenvalue  $\lambda$  of  $A$  such that  $p(\lambda) = \mu$ .

The famous Cayley–Hamilton theorem states that for a linear transformation  $A$  on  $\mathbf{C}^n$  and its characteristic polynomial  $q_A(z)$  as above, we have that

$$(1.110) \quad q_A(A) = 0.$$

It follows that

$$(1.111) \quad p(A) = 0$$

whenever  $p(z)$  is a polynomial which can be expressed as the product of the characteristic polynomial  $q_A(z)$  and another polynomial. This holds when  $p(z)$  vanishes at each eigenvalue of  $A$ , and to at least the same order as  $q_A$  does.

In particular, we have that  $A^n$  can be expressed as a linear combination of  $A^k$ ,  $1 \leq k \leq n-1$ , and the identity operator. By repeating this, every positive integer power of  $A$  can be expressed as a linear combination of  $A^k$ ,  $1 \leq k \leq n-1$ , and the identity operator. To put it another way, for each polynomial  $p(z)$  there is a polynomial  $\tilde{p}(z)$  of degree at most  $n-1$  such that  $p(A) = \tilde{p}(A)$ .

Also, the exponential of  $A$  can be expressed as  $p(A)$  for a polynomial  $p(z)$ . It is enough to choose  $p(z)$  so that it agrees with the exponential function at the eigenvalues of  $A$ , and to sufficiently high order. Notice in particular that the eigenvalues of  $\exp(A)$  are therefore all exponentials of eigenvalues of  $A$ .

We know that the exponential of a linear transformation is automatically invertible. Conversely, if  $B$  is an invertible linear transformation on  $\mathbf{C}^n$ , is there a linear transformation  $A$  on  $\mathbf{C}^n$  such that  $\exp(A) = B$ ? The answer is yes, and indeed one can take  $A = p(B)$ , where  $p(z)$  is a polynomial on  $\mathbf{C}$  which satisfies  $\exp(p(z)) = z$  at the eigenvalues of  $B$ , and to sufficiently high order.

Now let us consider the real case. We have seen that the determinant of the exponential of a linear transformation on  $\mathbf{R}^n$  is equal to the exponential of the trace of that linear transformation, and hence is a positive real number. This is a simple necessary condition for an invertible linear transformation on  $\mathbf{R}^n$  to be the exponential of another linear transformation.

Let  $A$  be any linear transformation on  $\mathbf{R}^n$ , and let  $\hat{A}$  denote the unique linear transformation on  $\mathbf{C}^n$  which agree with  $A$  on  $\mathbf{R}^n$ . To be more precise,  $\hat{A}$  is complex-linear, so that  $\hat{A}(iv) = i\hat{A}(v)$ , and this ensures that  $\hat{A}$  is determined by its action on vectors with real coordinates. Also,  $A$  and  $\hat{A}$  are associated to the same  $n \times n$  matrix with real entries, with respect to the standard bases of  $\mathbf{R}^n$  and  $\mathbf{C}^n$ , respectively.

For any polynomial  $p(x)$  with real coefficients, we can define  $p(A)$  in the usual manner, and it has the same basic properties as before. We can also think of  $p$  as a complex polynomial and consider  $p(\hat{A})$ , and it is easy to see that this is the same as the complex-linear transformation on  $\mathbf{C}^n$  induced by  $p(A)$ . In other words,

$$(1.112) \quad \widehat{p(A)} = p(\hat{A}).$$

If  $\lambda$  is a real number which is an eigenvalue of  $A$ , then  $\lambda$  is also an eigenvalue of  $\hat{A}$ , using the same eigenvector in fact, and conversely if  $\lambda$  is an eigenvalue of  $\hat{A}$  which is a real number too, then one can check that  $\lambda$  is an eigenvalue for  $A$ . However, in general there can be complex eigenvalues for  $\hat{A}$ , and one can check that if  $\lambda$  is an eigenvalue of  $\hat{A}$ , then so is the complex conjugate  $\bar{\lambda}$ , and with the same multiplicity as a zero of the characteristic polynomial  $q_{\hat{A}}$ . Notice that for  $x$  real the characteristic polynomial  $q_{\hat{A}}(x)$  of  $\hat{A}$  is the same as the real version for  $A$ ,

$$(1.113) \quad q_{\hat{A}}(x) = \det(xI - A),$$

and in particular the two polynomials have the same coefficients, which are real numbers.

Suppose that  $B$  is an invertible linear transformation on  $\mathbf{R}^n$ . If, for instance,  $B$  has no eigenvalues which are negative real numbers, then there are polynomials  $p(z)$  with real coefficients such that  $\exp(p(z)) = z$  to whatever order one might like at the eigenvalues of  $\hat{B}$ . Consequently,  $B = \exp(A)$  with  $A = p(B)$ , and where  $A$  is a linear transformation on  $\mathbf{R}^n$ .

This is certainly not the whole story however. Let us mention two basic examples of linear transformations with positive determinant and negative real eigenvalues which can and which cannot be represented as an exponential. We can look at this in terms of another correspondence between real and complex-linear transformations.

Namely, we can identify  $\mathbf{C}^n$  with  $\mathbf{R}^{2n}$  in the obvious way, with the real and imaginary parts of the  $n$  complex components of a vector in  $\mathbf{C}^n$  being the  $2n$  real components of the corresponding vector in  $\mathbf{R}^{2n}$ . If  $A$  is a linear transformation on  $\mathbf{C}^n$ , let us write  $A^\circ$  for the corresponding real-linear transformation on  $\mathbf{R}^{2n}$ . Notice that

$$(1.114) \quad (\alpha_1 A_1 + \alpha_2 A_2)^\circ = \alpha_1 A_1^\circ + \alpha_2 A_2^\circ$$

when  $\alpha_1, \alpha_2$  are real numbers and  $A_1, A_2$  are complex-linear transformations on  $\mathbf{C}^n$ , and that  $(A_1 A_2)^\circ = A_1^\circ A_2^\circ$ .

On  $\mathbf{R}^2$ , consider the linear transformation  $-I$ . This is a diagonalizable linear transformation with eigenvalue  $-1$  of multiplicity 2, and the determinant is equal to 1. This linear transformation is the exponential of another linear transformation on  $\mathbf{R}^2$ , because one can think of it as a complex-linear transformation on  $\mathbf{C}$ , and convert the realization as an exponential there to one on  $\mathbf{R}^2$ .

As a different example, suppose that  $B$  is a linear transformation on  $\mathbf{R}^2$  such that the two standard basis vectors  $e_1 = (1, 0)$  and  $e_2 = (0, 1)$  are eigenvectors with eigenvalues  $\lambda_1, \lambda_2$ , respectively, and where  $\lambda_1, \lambda_2$  are distinct negative real numbers. If  $B = \exp(A)$  for some real linear transformation  $A$  on  $\mathbf{R}^2$ , then  $A, B$  commute in particular, and it follows that  $A(e_1), A(e_2)$  are eigenvectors for  $B$  with eigenvalues  $\lambda_1, \lambda_2$ , respectively. Hence  $A(e_1), A(e_2)$  should be real multiples of  $e_1, e_2$ , this leads to a contradiction.

If  $A$  is any complex-linear transformation on  $\mathbf{C}^n$  and  $A^\circ$  is the corresponding

real-linear transformation on  $\mathbf{R}^{2n}$ , then

$$(1.115) \quad \det A^\circ = |\det A|^2,$$

i.e., the determinant of  $A^\circ$  as a real-linear transformation is equal to the absolute value squared of the determinant of  $A$  as a complex-linear transformation. This is not too difficult to show, starting with  $n = 1$ , for instance. In particular,  $A^\circ$  always has nonnegative determinant.

There is another well-known simple trick for expressing a positive power of a linear transformation as a linear combination of lower powers. Namely, if  $T$  is a linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , then there is a positive integer  $k \leq n^2$  such that  $T^k$  is a linear combination of the identity operator and  $T^j$ ,  $1 \leq j < k$ , simply because the vector space of linear transformations on  $\mathbf{R}^n$  or  $\mathbf{C}^n$  has dimension  $n^2$ . Of course the version of this from the Cayley–Hamilton theorem is more precise and explicit.

Suppose that  $A$  is a linear transformation on  $\mathbf{R}^n$ , and let  $\hat{A}$  denote the corresponding complex-linear transformation on  $\mathbf{C}^n$ . Of course

$$(1.116) \quad \|\hat{A}\|_{HS} = \|A\|_{HS},$$

since  $\hat{A}$  and  $A$  correspond to the same  $n \times n$  matrix of real numbers, and the norms in question are simply the square root of the sum of squares of these matrix entries. Moreover, one can check that

$$(1.117) \quad \|\hat{A}\|_{op} = \|A\|_{op},$$

where the left side refers to the operator norm of  $\hat{A}$  as a linear transformation on  $\mathbf{C}^n$ , and the right side refers to the operator norm of  $A$  as a linear transformation on  $\mathbf{R}^n$ .

Also,

$$(1.118) \quad \hat{A}^* = \widehat{(A^*)},$$

where the left side is the adjoint of  $\hat{A}$  as a linear transformation on  $\mathbf{C}^n$ , and the right side is the complex-linear transformation on  $\mathbf{C}^n$  induced by the adjoint of  $A$  as a real-linear transformation on  $\mathbf{R}^n$ . It follows that if  $A$  is an orthogonal linear transformation on  $\mathbf{R}^n$ , then  $\hat{A}$  is a unitary linear transformation. One can see this as well using the fact that a real or complex-linear transformation is orthogonal or unitary, respectively, if and only if it is invertible, has operator norm equal to 1, and its inverse has operator norm equal to 1.

If  $A$  is an orthogonal or unitary linear transformation on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , respectively, then

$$(1.119) \quad |\det A| = 1.$$

Indeed, in this case we have that

$$(1.120) \quad 1 = \det I = \det(AA^*) = (\det A)(\det A^*) = |\det A|^2.$$

Alternatively, one can show this using the fact that

$$(1.121) \quad |\det A| \leq \|A\|_{op}^n.$$

More precisely, if  $A$  is a linear transformation on  $\mathbf{C}^n$ , then  $\det A$  is the product of the eigenvalues of  $A$ , according to their multiplicities as zeros of the characteristic polynomial of  $A$ , and it is easy to see that

$$(1.122) \quad |\lambda| \leq \|A\|_{op}$$

for each eigenvalue of  $A$ . In the real case one can apply this argument to the induced complex-linear transformation on  $\mathbf{C}^n$ , which has the same determinant. As another argument, it is enough to check that

$$(1.123) \quad |\det A| \leq 1 \quad \text{when} \quad \|A\|_{op} \leq 1,$$

and for this one can observe that the sequence of linear transformations  $A^k$ ,  $k \geq 1$ , is bounded when  $\|A\|_{op} \leq 1$ , and hence that their determinants are bounded, and hence that the scalars  $(\det A)^k$  are bounded, which implies that  $|\det A| \leq 1$ .

Suppose that  $A$  is a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  which is anti-self-adjoint, so that

$$(1.124) \quad A^* = -A.$$

In this case  $\exp(A)$  is an orthogonal or unitary transformation, as appropriate. The adjoint of  $\exp(A)$  is equal to  $\exp(A^*)$ , which is the same as  $\exp(-A)$  in this case, which is the inverse of  $\exp(A)$ .

Let us consider next the question of when an orthogonal linear transformation on  $\mathbf{R}^n$  or a unitary transformation on  $\mathbf{C}^n$  can be expressed as the exponential of a self-adjoint linear transformation. To do this we digress a bit for some general matters about linear transformations. We begin with the complex case.

A linear transformation  $T$  on  $\mathbf{C}^n$  is said to be *normal* if  $T$  commutes with its adjoint, which is to say that

$$(1.125) \quad T^* T = T T^*.$$

We can write any linear transformation  $T$  on  $\mathbf{C}^n$  as  $T_1 + iT_2$ , where  $T_1, T_2$  are the self-adjoint linear transformations given by

$$(1.126) \quad T_1 = \frac{1}{2}(T + T^*), \quad T_2 = \frac{1}{2i}(T - T^*),$$

and the condition of normality is equivalent to saying that  $T_1, T_2$  commute. Note that unitary transformations are normal.

We already know that if  $B$  is a self-adjoint linear transformation, then there is an orthonormal basis of the underlying vector space consisting of eigenvectors of  $B$ . Given two self-adjoint linear transformations which commute, one can find an orthonormal basis consisting of vectors which are eigenvectors for both linear transformations. Conversely, for a fixed basis, any two linear transformations for which vectors in the basis are eigenvectors clearly commute with each other.

As a result, if  $T$  is a normal linear transformation on  $\mathbf{C}^n$ , then there is an orthonormal basis of  $\mathbf{C}^n$  consisting of eigenvectors of  $T$ . In particular this applies

to unitary transformations, for which the corresponding eigenvalues are complex numbers with modulus 1. As a result, if  $U$  is a unitary linear transformation on  $\mathbf{C}^n$ , then there is an anti-self-adjoint linear transformation  $A$  on  $\mathbf{C}^n$  such that  $\exp(A) = U$ , and indeed one can take  $A$  to be diagonalized by the same basis as for  $U$ , with imaginary eigenvalues.

Now let us consider the real case. For this we cannot use the trick of writing an anti-self-adjoint linear transformation as “ $i$ ” times a self-adjoint linear transformation. There are other things that we can do, however.

Thus we let  $A$  be an anti-self-adjoint linear transformation on  $\mathbf{R}^n$ . Notice that

$$(1.127) \quad \langle A(v), v \rangle = 0$$

for all vectors  $v \in \mathbf{R}^n$ , and that in particular a nonzero vector  $v$  in  $\mathbf{R}^n$  is an eigenvector for  $A$  only if the corresponding eigenvalue is equal to 0, so that  $v$  lies in the kernel of  $A$ . Also, for each vector  $w$  in  $\mathbf{R}^n$ ,  $A(w)$  is orthogonal to every vector in the kernel of  $A$ .

A basic trick to study an anti-self-adjoint linear transformation  $A$  is to consider  $A^2$ , which is self-adjoint and has the same kernel as  $A$  does. If  $v$  is a vector in  $\mathbf{R}^n$  which is an eigenvector  $A^2$  with eigenvalue  $\lambda$ , then  $A(v)$  is an eigenvector for  $A^2$  with eigenvalue  $\lambda$  too, and of course  $A^2(v)$  is a multiple of  $v$ . As a result, if  $\lambda$  is a negative real number which is an eigenvalue of  $A^2$ , then one can check that the corresponding eigenspace

$$(1.128) \quad \{v \in \mathbf{R}^n : A^2(v) = \lambda v\}$$

has *even* dimension.

If  $T$  is an orthogonal linear transformation on  $\mathbf{R}^n$ , then we can write  $T = T_1 + T_2$ , where  $T_1 = (T + T^*)/2$  is self-adjoint,  $T_2 = (T - T^*)/2$  is anti-self-adjoint,  $T_1, T_2$  commute, and

$$(1.129) \quad \langle T_1(v), T_2(v) \rangle = 0$$

for all  $v \in \mathbf{R}^n$ . If  $\lambda_1, \lambda_2$  are eigenvalues of  $T_1, T_2^2$  such that the joint eigenspace

$$(1.130) \quad \{v \in \mathbf{R}^n : T_1(v) = \lambda_1 v, T_2^2(v) = \lambda_2 v\}$$

is nontrivial, then either  $\lambda_2 = 0$  and  $\lambda_1 = \pm 1$ , or  $\lambda_2 < 0$ ,  $\lambda_1^2 - \lambda_2 = 1$ , and the joint eigenspace has even dimension. One can show that the parity of the number of times that  $\lambda_1 = -1$  is even or odd according to whether  $\det T$  is 1 or  $-1$ , and that when  $\det T = 1$ , there is an anti-self-adjoint linear transformation  $A$  on  $\mathbf{R}^n$  such that  $T = \exp(A)$ .

If  $A$  is a self-adjoint linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then  $\exp(A)$  is also self-adjoint, and in fact  $\exp(A)$  is positive-definite, because  $\exp(A) = B^2$  where  $B$  is the self-adjoint linear transformation  $\exp(A/2)$ . Conversely, every self-adjoint positive-definite linear transformation  $P$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  can be realized as  $\exp(A)$  for a self-adjoint linear transformation  $A$ . This can be seen using an orthonormal basis of eigenvectors for  $P$ , and indeed one can choose  $A$  so that the vectors in the same basis are eigenvectors for  $A$ .

Actually, if  $A$  is a self-adjoint linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then there is an orthonormal basis of eigenvectors for  $A$ , and of course these same vectors are eigenvectors for  $\exp(A)$ , with the eigenvalues for  $\exp(A)$  being the exponentials of the corresponding eigenvalues for  $A$ . As a result, for a given self-adjoint positive-definite linear transformation  $P$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , the self-adjoint linear transformation  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , respectively, such that  $\exp(A) = P$  is *unique*. This is analogous to the situation for the ordinary exponential function on real numbers, while in the complex case one can have different numbers or linear transformations whose exponentials are equal to each other.

There is a natural mapping from the group of invertible linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  onto the self-adjoint positive-definite linear transformations on the same space, given by

$$(1.131) \quad T \mapsto T T^*.$$

If  $T$  is an invertible linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and  $R$  is an orthogonal or unitary linear transformation on the same space, as appropriate, then  $T$  and  $T R$  are sent by the mapping just defined to the same positive-definite linear transformation, since

$$(1.132) \quad (T R)(T R)^* = T R R^* T^* = T T^*.$$

Conversely, if  $T, T'$  are invertible linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  such that  $T'(T')^* = T T^*$ , then there is an orthogonal or unitary linear transformation  $R$ , as appropriate, such that

$$(1.133) \quad T' = T R.$$

Also, every self-adjoint positive-definite linear transformation  $P$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  arises this manner, and in fact can be written in a unique manner as  $Q^2$  for a self-adjoint positive-definite linear transformation  $Q$ . If  $A$  is an invertible linear mapping on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then we get a nice action of  $A$  on the self-adjoint positive-definite linear transformations by the formula

$$(1.134) \quad P \mapsto A P A^*.$$

If  $A_1$  and  $A_2$  are two invertible linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  such that  $A_1 P A_1^* = A_2 P A_2^*$  for all self-adjoint positive-definite linear transformations  $P$ , then  $A_1 = A_2$ , and if  $P_1, P_2$  are two self-adjoint positive-definite linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then there is an invertible linear transformation  $A$  on the same space such that  $P_2 = A P_1 A^*$ .

## 2 Spaces of matrices

As before, we write  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  for the spaces of real and complex-linear mappings on  $\mathbf{R}^n$  and  $\mathbf{C}^n$ , respectively. We also write  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  for the *general linear groups* of invertible real and complex-linear transformations on



$\mathbf{R}^n$ ,  $\mathbf{C}^n$ , respectively. We can identify  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  with  $\mathbf{R}^{n^2}$ ,  $\mathbf{C}^{n^2}$  using the standard correspondence between linear transformations and matrices, and in this way we have that  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  are open subsets of  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$ , respectively.

The determinant can be viewed as a homogeneous polynomial of degree  $n$  on  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$ , and  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  can be described as the subsets of  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  defined by the condition

$$(2.1) \quad \det T \neq 0.$$

At the identity operator, the differential of the determinant can be identified with the trace, since

$$(2.2) \quad \left( \frac{d}{dr} \det(I + r A) \right)_{r=0} = \operatorname{tr} A$$

for any linear transformation  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . If  $T$  is an invertible linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  and  $A$  is another linear transformation on the same space, then the differential of the determinant at  $T$  in the direction  $A$  can be expressed as

$$(2.3) \quad d(\det)_T(A) = (\det T) \operatorname{tr}(T^{-1} A),$$

since

$$(2.4) \quad \left( \frac{d}{dr} \det(T + r A) \right)_{r=0} = (\det T) \left( \frac{d}{dr} \det(I + r T^{-1} A) \right)_{r=0}$$

$$(2.5) \quad = (\det T) \operatorname{tr}(T^{-1} A).$$

We write  $SL(\mathbf{R}^n)$ ,  $SL(\mathbf{C}^n)$  for the subgroups of  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  consisting of linear transformations with determinant equal to 1. These are nice submanifolds of  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ , because the differential of the determinant is not equal to 0 at any point in  $SL(\mathbf{R}^n)$ ,  $SL(\mathbf{C}^n)$ , or in  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ , for that matter. Also we have the maps

$$(2.6) \quad T \mapsto (\det T)^{-1/n} T$$

from invertible linear transformations on  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  to linear transformations with determinant 1, at least if we restrict our attention to  $T$ 's with  $\det T > 0$  in the real case and  $T$ 's with  $\det T$  in a nice region in  $\mathbf{C}$  that contains 1 and on which  $z^{-1/n}$  can be defined in the complex case.

If  $T$  is an element of  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$ , then the space of tangent vectors to  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  at  $T$ , as appropriate, can be identified with  $\mathcal{L}(\mathbf{R}^n)$  or  $\mathcal{L}(\mathbf{C}^n)$ , as appropriate, since  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  are open subsets of  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$ , respectively. If  $T$  is an element of  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$ , then the space of tangent vectors to  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$  at  $T$  is equal to the space of  $A$  in  $\mathcal{L}(\mathbf{R}^n)$  or  $\mathcal{L}(\mathbf{C}^n)$  such that

$$(2.7) \quad \operatorname{tr}(T^{-1} A) = 0,$$

as appropriate. In other words, these are the tangent vectors to  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  at  $T$ , respectively, which lie in the kernel of the differential of the determinant function at  $T$ .

An important mapping on  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  is the one defined by

$$(2.8) \quad F(T) = T^{-1},$$

and which also sends  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$  to itself, as appropriate. For each invertible linear transformation  $T$  the differential of this mapping at  $T$  can be expressed as

$$(2.9) \quad dF_T(A) = -T^{-1} A T^{-1},$$

which is to say that

$$(2.10) \quad \left( \frac{d}{dr} (T + r A)^{-1} \right)_{r=0} = -T^{-1} A T^{-1}.$$

Indeed,

$$\begin{aligned} (I + r T^{-1} A)^{-1} &= (I + r T^{-1} A)^{-1} T^{-1} \\ &= (I - r T^{-1} A + O(r^2)) T^{-1} = T^{-1} - r T^{-1} A T^{-1} + O(r^2). \end{aligned}$$

If  $T$  is an invertible linear transformation on  $\mathbf{R}^n$  and  $A, B$  are linear transformations on  $\mathbf{R}^n$ , then set

$$(2.12) \quad \langle A, B \rangle_T = \text{tr}(T^{-1} A T^{-1} B).$$

This is a symmetric bilinear form in  $A, B$  for each  $T$ , which is to say that  $\langle A, B \rangle_T$  is a linear function of  $A$  for each  $B$  and  $T$ , a linear function of  $B$  for each  $A$  and  $T$ , and that in fact

$$(2.13) \quad \langle A, B \rangle_T = \langle B, A \rangle_T$$

for all  $A, B, T$ , so that linearity in  $A$  and  $B$  are equivalent to each other. Moreover, this bilinear form is nondegenerate, which means that for each  $T$  and for each  $A \neq 0$  there is a  $B$  such that  $\langle A, B \rangle_T \neq 0$ .

In the complex case, if  $T$  is an invertible linear transformation on  $\mathbf{C}^n$  and  $A, B$  are linear transformations on  $\mathbf{C}^n$ , then

$$(2.14) \quad \text{tr}(T^{-1} A T^{-1} B)$$

is a complex-valued symmetric bilinear form in  $A, B$  for each  $T$  which is nondegenerate. To get a real-valued quantity, we set

$$(2.15) \quad \langle A, B \rangle_T = \text{Re tr}(T^{-1} A T^{-1} B),$$

i.e., we take the real part of the trace. This is still real-bilinear, which means that it is real-linear in each of  $A$  and  $B$ , and symmetric and nondegenerate.

Notice that  $\langle A, B \rangle_T$  depends smoothly on  $T$  for  $T$  in  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$ , as appropriate. As a result,  $\langle A, B \rangle_T$  is said to define a semi-Riemannian structure, also known as a pseudo-Riemannian structure or a Riemannian structure with signature, on  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$ , as appropriate. On  $GL(\mathbf{C}^n)$ , if we did

not take the real part, then we would have a holomorphic semi-Riemannian structure.

At points  $T$  in  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$ , we can restrict our attention to  $A, B$  which are in the tangent space of  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$  at  $T$ , as appropriate. Explicitly, this means that we restrict our attention to  $A, B$  such that

$$(2.16) \quad \text{tr}(T^{-1} A) = \text{tr}(T^{-1} B) = 0.$$

This leads to semi-Riemannian structures on  $SL(\mathbf{R}^n)$ ,  $SL(\mathbf{C}^n)$ , and a holomorphic semi-Riemannian structure on  $SL(\mathbf{C}^n)$  if we do not take the real part.

For each linear transformation  $Z$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , define linear transformations  $\lambda_Z, \rho_Z$  on  $\mathcal{L}(\mathbf{R}^n)$  or on  $\mathcal{L}(\mathbf{C}^n)$ , respectively, by

$$(2.17) \quad \lambda_Z(T) = Z T, \quad \rho_Z(T) = T Z,$$

which is to say that  $\lambda_Z, \rho_Z$  correspond to left and right multiplication by  $Z$ . These are linear transformations, and in particular their differentials are given by themselves,

$$(2.18) \quad (d\lambda_Z)_T(A) = \lambda_Z(A), \quad (d\rho_Z)_T(A) = \rho_Z(A)$$

for all linear transformations  $T, A$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , as appropriate. If  $Z$  is an invertible linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then  $\lambda_Z, \rho_Z$  map  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  onto itself, as appropriate, and if

$$(2.19) \quad \det Z = 1,$$

then  $\lambda_Z, \rho_Z$  map  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$  onto itself, as appropriate.

If  $Z$  is an invertible linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then the mappings  $\lambda_Z, \rho_Z$  on  $GL(\mathbf{R}^n)$  or on  $GL(\mathbf{C}^n)$  preserve the semi-Riemannian structure  $\langle \cdot, \cdot \rangle_T$ . In other words, if  $T$  is an element of  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  and  $A, B$  are elements of  $\mathcal{L}(\mathbf{R}^n)$  or  $\mathcal{L}(\mathbf{C}^n)$ , as appropriate, which we view as tangent vectors to  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  at  $T$ , then

$$(2.20) \quad \langle (d\lambda_Z)_T(A), (d\lambda_Z)_T(B) \rangle_{\lambda_Z(T)} = \langle A, B \rangle_T$$

and

$$(2.21) \quad \langle (d\rho_Z)_T(A), (d\rho_Z)_T(B) \rangle_{\rho_Z(T)} = \langle A, B \rangle_T.$$

This is easy to verify. By restriction, if  $\det Z = 1$ , so that  $\lambda_Z, \rho_Z$  can be viewed as defining mappings on  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$ , as appropriate, then  $\lambda_Z, \rho_Z$  preserve the restriction of our semi-Riemannian structures to  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$ .

One can also check that  $F(T) = T^{-1}$  preserves the semi-Riemannian structures on  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ . That is, if  $T$  is an element of  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$ , and if  $A, B$  are elements of  $\mathcal{L}(\mathbf{R}^n)$  or  $\mathcal{L}(\mathbf{C}^n)$ , respectively, which we can view as tangent vectors to  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  at  $T$ , then

$$(2.22) \quad \langle (dF)_T(A), (dF)_T(B) \rangle_{F(T)} = \langle A, B \rangle_T.$$

Also,  $SL(\mathbf{R}^n)$  and  $SL(\mathbf{C}^n)$  are invariant under  $F(T) = T^{-1}$ , and the restriction of the semi-Riemannian structure to  $SL(\mathbf{R}^n)$ ,  $SL(\mathbf{C}^n)$  is preserved by  $F$ , since this holds on  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ .

In the complex case, let us note that  $\lambda_Z$ ,  $\rho_Z$  are complex-linear transformations, and  $F(T) = T^{-1}$  is a holomorphic transformation, and that they preserve the holomorphic version of the semi-Riemannian structure on  $GL(\mathbf{C}^n)$  and  $SL(\mathbf{C}^n)$ .

Of course we can define a flat semi-Riemannian metric on  $\mathcal{L}(\mathbf{R}^n)$  by saying that if  $T$ ,  $A$ ,  $B$  are linear transformations on  $\mathbf{R}^n$ , where we think of  $A$ ,  $B$  as tangent vectors to  $\mathcal{L}(\mathbf{R}^n)$  at  $T$ , then the inner product of these two tangent vectors associated to  $T$  is given by

$$(2.23) \quad \text{tr}(AB).$$

In the complex case the same formula defines a holomorphic semi-Riemannian structure on  $\mathcal{L}(\mathbf{C}^n)$ , and to get an ordinary semi-Riemannian structure one should take the real part. That (2.23) does not depend on  $T$  reflects the fact that these semi-Riemannian structures are flat. Of course we can restrict these semi-Riemannian structures to the subspaces  $\mathcal{L}_0(\mathbf{R}^n)$ ,  $\mathcal{L}_0(\mathbf{C}^n)$  of linear transformations with trace equal to 0.

The exponential mapping defines a mapping from  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  to  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  respectively, sending 0 to  $I$  and with

$$(2.24) \quad d\exp_0(A) = A.$$

In particular, the standard flat metric at 0 on  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  corresponds exactly to the semi-Riemannian structure on  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  at  $I$ , respectively, under the differential of the exponential mapping at 0. In fact,

$$(2.25) \quad \begin{aligned} \langle d\exp_T(A), d\exp_T(B) \rangle_{\exp T} \\ = \text{tr}(\exp(-T))(d\exp_T(A))(\exp(-T))(d\exp_T(B)) \end{aligned}$$

agrees with

$$(2.26) \quad \text{tr } AB$$

to another term in the Taylor expansion, which is to say up to terms of order  $O(\|T\|_{op}^2)$ . In other words, using  $\exp(-T) = I - T + O(\|T\|_{op}^2)$ ,

$$(2.27) \quad d\exp_T(A) = A + \frac{1}{2}(TA + AT) + O(\|T\|_{op}^2),$$

and similarly for  $B$ , one can check that the terms in (2.25) with no  $T$ 's reduce to  $\text{tr}(AB)$ , and that the terms with exactly one  $T$  cancel out. In the complex case, let us note that the exponential mapping is holomorphic, and one has the analogous statement about the holomorphic semi-Riemannian metrics on  $\mathcal{L}(\mathbf{C}^n)$  and  $GL(\mathbf{C}^n)$  agreeing at 0 up to terms of order  $O(\|T\|_{op}^2)$ .

As a consequence, for each linear transformation  $A$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ ,  $\exp(tA)$  satisfies the equation for geodesics at  $t = 0$ . This is because  $tA$  is simply a

straight line in  $\mathcal{L}(\mathbf{R}^n)$  or  $\mathcal{L}(\mathbf{C}^n)$ , as appropriate, and thus satisfies the equation for geodesics there with respect to the flat semi-Riemannian structure being used, and because the exponential mapping takes the flat semi-Riemannian structures on  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  around 0 to the semi-Riemannian structures on  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  around  $I$  to sufficient precision, as in the preceding paragraph. In fact, it follows that  $\exp(tA)$  satisfies the equation for geodesics for all  $t$ , because one can use the invariance of the semi-Riemannian metrics on  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  under ordinary translations and the invariance of the semi-Riemannian structures on  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  under left and right multiplication by invertible linear transformations to reduce the case of general  $t$  to  $t = 0$ . In the complex case, one can take  $t$  to be a complex parameter, and say that  $\exp(tA)$  is a holomorphic geodesic in  $GL(\mathbf{C}^n)$  with respect to the holomorphic semi-Riemannian structure as before.

If  $A, Y$  are linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  with  $Y$  invertible, then  $Y \exp(tA)$  defines a geodesic in  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$ , as appropriate, again using the invariance of the semi-Riemannian structures that we have defined. This is equivalent to saying that if  $B, Y$  are linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then  $\exp(tB)Y$  defines a geodesic in  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$ , as appropriate, since  $Y \exp(tA) = \exp(tB)Y$  with  $B = YAY^{-1}$ , and anyway our semi-Riemannian structures on the general linear groups are invariant under both left and right multiplications. This accounts for all of the geodesics, because the equation for geodesics are described by a second-order differential equation, and thus a geodesic is characterized by a point that it passes through and the tangent vector corresponding to its derivative at that point.

The preceding discussion can also be applied to the restriction of the exponential mapping to the subspaces  $\mathcal{L}_0(\mathbf{R}^n)$ ,  $\mathcal{L}_0(\mathbf{C}^n)$  of  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$  taking values in the subgroups  $SL(\mathbf{R}^n)$ ,  $SL(\mathbf{C}^n)$  of  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ , using the restrictions of the corresponding semi-Riemannian metrics. In particular,  $SL(\mathbf{R}^n)$ ,  $SL(\mathbf{C}^n)$  are *totally geodesic* submanifolds of  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ . That is, a geodesic in  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  which passes through  $SL(\mathbf{R}^n)$  or  $SL(\mathbf{C}^n)$  and is tangent to the special linear group at the point of intersection stays in the special linear group. Note that  $SL(\mathbf{C}^n)$  is a complex submanifold of  $GL(\mathbf{C}^n)$ .

Fix a positive integer  $n$ , and let  $\mathcal{F}$  be a *flag* in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , which is to say a family  $L_1, L_2, \dots, L_k$  of distinct nontrivial proper linear subspaces of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  with

$$(2.28) \quad L_1 \subseteq L_2 \subseteq \dots \subseteq L_k.$$

Thus  $k$  is a positive integer strictly less than  $n$ , called the *length* of the flag. It may be that  $k = 1$ , so that the flag consists of a single nontrivial proper linear subspace.

If  $\mathcal{F}$  is a flag in  $\mathbf{R}^n$  or in  $\mathbf{C}^n$ , then we write  $\mathcal{L}_{\mathcal{F}}(\mathbf{R}^n)$  or  $\mathcal{L}_{\mathcal{F}}(\mathbf{C}^n)$  for the space of linear transformations  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, such that  $A(L_j) \subseteq L_j$  for each of the linear subspaces  $L_j$  in the flag. Thus  $\mathcal{L}_{\mathcal{F}}(\mathbf{R}^n)$ ,  $\mathcal{L}_{\mathcal{F}}(\mathbf{C}^n)$  are themselves linear subspaces of  $\mathcal{L}(\mathbf{R}^n)$ ,  $\mathcal{L}(\mathbf{C}^n)$ , respectively, which are also closed under taking products of linear transformations. By using bases for  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, which are suitably adapted to the flag  $\mathcal{F}$ , one can also characterize

the linear transformations in  $\mathcal{L}_{\mathcal{F}}(\mathbf{R}^n)$  or  $\mathcal{L}_{\mathcal{F}}(\mathbf{C}^n)$ , as appropriate, in terms of matrices with certain entries equal to 0.

Similarly, if  $\mathcal{F}$  is a flag in  $\mathbf{R}^n$  or in  $\mathbf{C}^n$ , then we write  $GL_{\mathcal{F}}(\mathbf{R}^n)$  or  $GL_{\mathcal{F}}(\mathbf{C}^n)$  for the space of invertible linear transformations  $T$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, such that  $T(L_j) = L_j$  for each linear subspace  $L_j$  in the flag. This is equivalent to saying that

$$(2.29) \quad GL_{\mathcal{F}}(\mathbf{R}^n) = GL(\mathbf{R}^n) \cap \mathcal{L}_{\mathcal{F}}(\mathbf{R}^n)$$

and

$$(2.30) \quad GL_{\mathcal{F}}(\mathbf{C}^n) = GL(\mathbf{C}^n) \cap \mathcal{L}_{\mathcal{F}}(\mathbf{C}^n).$$

Furthermore, let us put

$$(2.31) \quad SL_{\mathcal{F}}(\mathbf{R}^n) = SL(\mathbf{R}^n) \cap \mathcal{L}_{\mathcal{F}}(\mathbf{R}^n)$$

and

$$(2.32) \quad SL_{\mathcal{F}}(\mathbf{C}^n) = SL(\mathbf{C}^n) \cap \mathcal{L}_{\mathcal{F}}(\mathbf{C}^n).$$

The exponential mapping can be restricted to  $\mathcal{L}_{\mathcal{F}}(\mathbf{R}^n)$  or  $\mathcal{L}_{\mathcal{F}}(\mathbf{C}^n)$  to get a mapping into  $GL_{\mathcal{F}}(\mathbf{R}^n)$  or  $GL_{\mathcal{F}}(\mathbf{C}^n)$ , as appropriate. One can restrict a bit further to linear transformations in  $\mathcal{L}_{\mathcal{F}}(\mathbf{R}^n)$  or  $\mathcal{L}_{\mathcal{F}}(\mathbf{C}^n)$  with trace equal to 0, which the exponential mapping sends to linear transformations in  $SL_{\mathcal{F}}(\mathbf{R}^n)$  or  $SL_{\mathcal{F}}(\mathbf{C}^n)$ . As before, one can account for all of the geodesics in  $GL_{\mathcal{F}}(\mathbf{R}^n)$ ,  $GL_{\mathcal{F}}(\mathbf{C}^n)$  or in  $SL_{\mathcal{F}}(\mathbf{R}^n)$ ,  $SL_{\mathcal{F}}(\mathbf{C}^n)$  using exponentials, and these define totally geodesic submanifolds of  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ , respectively.

Let us write  $\mathcal{S}(bfR^n)$ ,  $\mathcal{S}(\mathbf{C}^n)$  for the real vector spaces of self-adjoint linear transformations on  $\mathbf{R}^n$ ,  $\mathbf{C}^n$ , respectively, and  $\mathcal{S}_+(\mathbf{R}^n)$ ,  $\mathcal{S}_+(\mathbf{C}^n)$  for their open cones of positive-definite linear transformations. These subsets are invariant under the transformation  $F(T) = T^{-1}$ , and also under the action

$$(2.33) \quad T \mapsto Z^* T Z$$

for each  $Z$  in  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$ , as appropriate. In fact, this action is *transitive*, which is to say that for each  $T_1$ ,  $T_2$  in  $\mathcal{S}_+(\mathbf{R}^n)$  or in  $\mathcal{S}_+(\mathbf{C}^n)$  there is a  $Z$  in  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$ , as appropriate, such that  $Z^* T_1 Z = T_2$ .

The restriction of our semi-Riemannian structures on  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  to  $\mathcal{S}_+(\mathbf{R}^n)$ ,  $\mathcal{S}_+(\mathbf{C}^n)$ , respectively, are *Riemannian* metrics, which is to say that they are positive definite. Of course these Riemannian metrics are invariant under the transformations preserving  $\mathcal{S}_+(\mathbf{R}^n)$ ,  $\mathcal{S}_+(\mathbf{C}^n)$  mentioned in the previous paragraph. The exponential mapping sends  $\mathcal{S}(\mathbf{R}^n)$ ,  $\mathcal{S}(\mathbf{C}^n)$  onto  $\mathcal{S}_+(\mathbf{R}^n)$ ,  $\mathcal{S}_+(\mathbf{C}^n)$ , respectively, and the geodesics in  $\mathcal{S}_+(\mathbf{R}^n)$ ,  $\mathcal{S}_+(\mathbf{C}^n)$  through  $I$  are exactly the curves  $\exp(tA)$ , with  $A$  in  $\mathcal{S}(\mathbf{R}^n)$ ,  $\mathcal{S}(\mathbf{C}^n)$ , respectively. The geodesics through a point  $T = Z^* Z$  are of the form  $Z^* \exp(tA) Z$ .

The orthogonal and unitary groups on  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  are denoted  $O(\mathbf{R}^n)$  and  $U(\mathbf{C}^n)$  and consist of the invertible linear transformations  $T$  which are orthogonal or unitary, respectively, which is to say that

$$(2.34) \quad T^* T = T T^* = I.$$

It is enough to have  $T^*T = I$ , and at a point  $T$  in  $O(\mathbf{R}^n)$  or  $U(\mathbf{C}^n)$  the tangent space to the orthogonal or unitary group consists of the linear transformations  $A$  on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, such that

$$(2.35) \quad T^*A + A^*T = 0.$$

Let us note that the orthogonal and unitary groups are compact smooth submanifolds of the vector spaces of all linear transformations on  $\mathbf{R}^n$ ,  $\mathbf{C}^n$ , and of  $GL(\mathbf{R}^n)$ ,  $GL(\mathbf{C}^n)$  in particular.

Again we can restrict our semi-Riemannian structures from  $GL(\mathbf{R}^n)$  or  $GL(\mathbf{C}^n)$  to the submanifolds given by the orthogonal and unitary groups, respectively. Now these restricted structures are negative-definite, so that their negatives are Riemannian metrics. Using the group structure we again have the mappings  $\lambda_Z(T) = ZT$  and  $\rho_Z(T) = TZ$  which send the orthogonal and unitary groups to themselves as long as  $Z$  also lies in the orthogonal or unitary group, and also the mapping  $F(T) = T^{-1}$  takes the orthogonal and unitary groups to themselves as well. The negative Riemannian metrics on the orthogonal and unitary groups are preserved by these transformations. If  $A$  is an anti-self-adjoint linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , then  $\exp(tA)$  defines a geodesic in  $O(\mathbf{R}^n)$  or  $U(\mathbf{C}^n)$ , as appropriate, and this accounts for all geodesics in the orthogonal and unitary groups through  $I$ , and hence for all geodesics if one also takes into account the left or right translation mappings  $\lambda_Z$ ,  $\rho_Z$ , as before.

In these various cases one can restrict further to linear transformations with determinant equal to 1. Let us write  $\mathcal{M}(\mathbf{R}^n)$ ,  $\mathcal{M}(\mathbf{C}^n)$  for the hypersurfaces in  $\mathcal{S}_+(\mathbf{R}^n)$ ,  $\mathcal{S}_+(\mathbf{C}^n)$  consisting of linear transformations with determinant equal to 1, and  $SO(\mathbf{R}^n)$  and  $SU(\mathbf{C}^n)$  for the special orthogonal and unitary groups on  $\mathbf{R}^n$  and  $\mathbf{C}^n$ , which are the subgroups of  $O(\mathbf{R}^n)$ ,  $U(\mathbf{C}^n)$  determined by the condition that the determinant of the corresponding linear transformation be equal to 1. There are similar considerations as before concerning tangent vectors, Riemannian structures, geodesics, and so on.

### 3 Some geometric situations

As usual,  $\mathbf{Z}$  denotes the integers, and  $\mathbf{Z}^n$  consists of  $n$ -tuples of integers. Sometimes we might refer to  $\mathbf{Z}^n$  as the *standard integer lattice* in  $\mathbf{R}^n$ . If we say that  $L$  is a *lattice* in  $\mathbf{R}^n$ , then we mean that there is an invertible linear transformation  $A$  on  $\mathbf{R}^n$  such that

$$(3.1) \quad L = A(\mathbf{Z}^n).$$

If  $L$  is a lattice in  $\mathbf{R}^n$ , then we can form the quotient space  $\mathbf{R}^n/L$ . That is, two vectors  $x, y$  in  $\mathbf{R}^n$  are identified in the quotient if their difference  $x - y$  lies in  $L$ . In particular, we get a canonical quotient mapping

$$(3.2) \quad q : \mathbf{R}^n \rightarrow \mathbf{R}^n/L$$

which sends a vector  $x$  in  $\mathbf{R}^n$  to the corresponding element of the quotient.

Now, with respect to ordinary vector addition,  $\mathbf{R}^n$  is an abelian group, and a lattice  $L$  is a subgroup of  $\mathbf{R}^n$ . We can think of the quotient  $\mathbf{R}^n/L$  as a quotient in the sense of group theory. The quotient is an abelian group under addition, and the canonical quotient mapping is a group homomorphism.

We can also look at the quotient  $\mathbf{R}^n/L$  in terms of topology. Namely, it inherits a topology from the one on  $\mathbf{R}^n$  so that the canonical quotient mapping is an open continuous mapping, which means that both images and inverse images of open sets are open sets, and indeed the canonical quotient mapping is a nice covering mapping, so that for every point  $x$  in  $\mathbf{R}^n$  there is a neighborhood  $U$  of  $x$  in  $\mathbf{R}^n$  such that the restriction of  $q$  to  $U$  is a homeomorphism from  $U$  onto the open set  $q(U)$  in  $\mathbf{R}^n/L$ . For that matter we can think of  $\mathbf{R}^n/L$  as a smooth manifold, with the quotient mapping  $q$  as a smooth mapping which is a local diffeomorphism.

Suppose that  $L_1, L_2$  are lattices in  $\mathbf{R}^n$ , and let

$$(3.3) \quad q_1 : \mathbf{R}^n \rightarrow \mathbf{R}^n/L_1, \quad q_2 : \mathbf{R}^n \rightarrow \mathbf{R}^n/L_2$$

be the corresponding canonical quotient mappings. If  $A$  is an invertible linear transformation on  $\mathbf{R}^n$  such that

$$(3.4) \quad A(L_1) = L_2,$$

then we get an induced mapping

$$(3.5) \quad \hat{A} : \mathbf{R}^n/L_1 \rightarrow \mathbf{R}^n/L_2.$$

This mapping is a group isomorphism and a homeomorphism, and even a diffeomorphism, which satisfies the obvious compatibility condition with the corresponding canonical quotient mappings  $q_1, q_2$ , namely  $q_1 \circ A = \hat{A} \circ q_2$ .

When  $n = 1$ , one can consider the lattice  $2\pi\mathbf{Z}$  consisting of integer multiples of  $2\pi$ , and it is customary to identify  $\mathbf{R}/2\pi\mathbf{Z}$  with the unit circle  $\mathbf{T}$  in the complex numbers  $\mathbf{C}$ ,

$$(3.6) \quad \mathbf{T} = \{z \in \mathbf{C} : |z| = 1\},$$

where  $|z|$  denotes the usual modulus of  $z \in \mathbf{C}$ ,  $|z| = (x^2 + y^2)^{1/2}$  when  $z = x + iy$ ,  $x, y \in \mathbf{R}$ . More precisely,  $\exp(it)$  is an explicit version of the canonical quotient mapping from  $\mathbf{R}/2\pi\mathbf{Z}$  onto  $\mathbf{T}$  with respect to this identification, which is a local diffeomorphism and a group homomorphism using the group structure of multiplication on  $\mathbf{T}$ . In general, we can identify  $\mathbf{R}^n/2\pi\mathbf{Z}^n$  with  $\mathbf{T}^n$ , the  $n$ -fold Cartesian product of  $\mathbf{T}$ , where  $2\pi\mathbf{Z}^n$  denotes the lattice of points whose coordinates are all integer multiples of  $2\pi$ .

Suppose that  $L$  is a lattice in  $\mathbf{R}^n$ . Also let  $A$  be an invertible linear mapping on  $\mathbf{R}^n$  such that  $A(2\pi\mathbf{Z}^n) = L$ . Thus  $\hat{A}$  is a group isomorphism and a diffeomorphism from  $\mathbf{R}^n/2\pi\mathbf{Z}^n \cong \mathbf{T}^n$  onto  $\mathbf{R}^n/L$ .

There is a more precise way to look at the quotient of  $\mathbf{R}^n$  by a lattice, which is to say that the quotient space has a kind of local affine structure. That is, there is a local affine structure in which the canonical quotient mapping is considered to be locally affine, and which permits one to say when a curve in the



quotient is locally a straight line segment, like an arc on a line, and when it has locally constant speed, etc. If  $L_1, L_2$  are lattices in  $\mathbf{R}^n$  and  $A$  is an invertible linear mapping on  $\mathbf{R}^n$  such that  $A(L_1) = L_2$ , then the induced mapping  $\hat{A}$  from  $\mathbf{R}^n/L_1$  onto  $\mathbf{R}^n/L_2$  preserves this local affine structure on the quotient spaces.

There is an even more precise way to look at the quotient  $\mathbf{R}^n/L$  of  $\mathbf{R}^n$  by a lattice  $L$ , which is that it has a local flat geometric structure, induced from the one on  $\mathbf{R}^n$ . With respect to this structure one can make local measurements of lengths, volumes, and angles, like the length of a curve, the angle at which two curves meet at a point, or the volume of a nice subset. In technical terms this can be seen as a *Riemannian metric*.

In particular, one can define the volume of such a quotient  $\mathbf{R}^n/L$ , where the volume of  $\mathbf{R}^n/\mathbf{Z}^n$  is equal to 1, and the volume of  $\mathbf{R}^n/2\pi\mathbf{Z}^n$  is equal to  $(2\pi)^n$ . In general, if  $L_1, L_2$  are lattices in  $\mathbf{R}^n$  and  $A$  is an invertible linear transformation on  $\mathbf{R}^n$  such that  $A(L_1) = L_2$ , then the volume of  $\mathbf{R}^n/L_2$  is equal to  $|\det A|$  times the volume of  $\mathbf{R}^n/L_1$ , and more generally if  $E$  is a nice subset of  $\mathbf{R}^n/L_1$ , then the volume of  $\hat{A}(E)$  in  $\mathbf{R}^n/L_2$  is equal to  $|\det A|$  times the volume of  $A$  in  $\mathbf{R}^n/L_1$ . This is a variant of the fact that on  $\mathbf{R}^n$  a linear transformation  $A$  distorts volumes by a factor of  $|\det A|$ , where  $\det A$  denotes the determinant of  $A$ .

Suppose that  $L_1, L_2$  are lattices in  $\mathbf{R}^n$ , and that  $T$  is an invertible linear transformation on  $\mathbf{R}^n$  such that  $T(L_1) = L_2$ . Recall that  $T$  is an *orthogonal transformation* on  $\mathbf{R}^n$  if  $T$  is invertible with inverse given by the adjoint, also known as the transpose, of  $T$ , and that this is equivalent to saying that  $T$  preserves the standard norm of vectors in  $\mathbf{R}^n$ , and the standard inner product of vectors in  $\mathbf{R}^n$ . In other words, orthogonal transformations on  $\mathbf{R}^n$  are linear mappings which preserve the geometry in  $\mathbf{R}^n$ , and for the lattices  $L_1, L_2$  and the quotients of  $\mathbf{R}^n$  by them we have that the induced mapping  $\hat{T}$  from  $\mathbf{R}^n/L_1$  onto  $\mathbf{R}^n/L_2$  preserves the geometry as well.

In short, quotients of  $\mathbf{R}^n$  by lattices are the same in terms of group structure, topological and even smooth structure, and affine structure, and not in general for more precise geometry. The volume of the quotient space is one basic parameter that one can consider. It is also interesting to look at closed curves in the quotient which are locally flat, their lengths, the angles at which they meet, and so on.

We can consider lattices in  $\mathbf{C}^n$  as well. In this regard we can identify  $\mathbf{C}^n$  with  $\mathbf{R}^{2n}$  in the usual manner, so that the real and imaginary parts of the  $n$  components of an element of  $\mathbf{C}^n$  give rise to the  $2n$  components of an element of  $\mathbf{R}^{2n}$ , and then define a lattice in  $\mathbf{C}^n$  to be a lattice in  $\mathbf{R}^{2n} \sim \mathbf{C}^n$ . We write  $\mathbf{Z}[i]$  for the *Gaussian integers*, which are complex numbers of the form  $a + ib$ , where  $a, b$  are integers, and  $(\mathbf{Z}[i])^n$  for the lattice in  $\mathbf{C}^n$  consisting of  $n$ -tuples of Gaussian integers, which is also called the standard integer lattice in  $\mathbf{C}^n$ .

If  $L$  is a lattice in  $\mathbf{C}^n$ , then the quotient  $\mathbf{C}^n/L$  inherits a complex structure from  $\mathbf{C}^n$ . This means in particular that the tangent spaces of the quotient are complex vector spaces, just as they are for  $\mathbf{C}^n$ . If  $L_1, L_2$  are lattices in  $\mathbf{C}^n$  and  $A$  is an invertible complex-linear transformation on  $\mathbf{C}^n$  such that  $A(L_1) = L_2$ ,

then  $A$  induces a mapping  $\hat{A}$  from  $\mathbf{C}^n/L_1$  to  $\mathbf{C}^n/L_2$  which preserves this complex structure.

We can combine the complex and Riemannian structures and consider *Hermitian structures*. Basically this means looking at correspondences between lattices in  $\mathbf{C}^n$  which come from unitary mappings on  $\mathbf{C}^n$ . If  $L_1, L_2$  are lattices in  $\mathbf{C}^n$  and  $T$  is a unitary mapping on  $\mathbf{C}^n$  such that  $T(L_1) = L_2$ , then the induced mapping  $\hat{T}$  from  $\mathbf{C}^n/L_1$  to  $\mathbf{C}^n/L_2$  preserves both the complex and Riemannian structures.

Let us focus on complex structures for a moment. It will be convenient to write  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  for the space of real-linear mappings from  $\mathbf{R}^m$  to  $\mathbf{C}^n$ . The complex structure on  $\mathbf{C}^n$  is still relevant for this space, in that  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  is naturally a complex vector space, because one can multiply elements of  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  by  $i$ . One can also describe these linear transformations by  $m \times n$  matrices of complex numbers in the usual manner, using the standard bases for  $\mathbf{R}^m$  and  $\mathbf{C}^n$ .

Let us write  $\mathcal{L}^*(\mathbf{R}^m, \mathbf{C}^n)$  for the subset of  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$  consisting of linear transformations whose kernels are trivial, at least when  $m \leq 2n$ , so that this is possible. Using the usual Euclidean topology for  $\mathcal{L}(\mathbf{R}^m, \mathbf{C}^n)$ ,  $\mathcal{L}^*(\mathbf{R}^m, \mathbf{C}^n)$  is an open set. When  $m = 2n$ ,  $\mathcal{L}^*(\mathbf{R}^m, \mathbf{C}^n)$  consists of the invertible real-linear transformations from  $\mathbf{R}^m$  onto  $\mathbf{C}^n$ , and a lattice in  $\mathbf{C}^n$  is the image of  $\mathbf{Z}^{2n}$  under an element of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ .

Now let us look at general lattices in  $\mathbf{C}^n$ , under the equivalence relation in which two lattices  $L_1, L_2$  are considered to be equivalent if there is an invertible complex-linear transformation  $A$  on  $\mathbf{C}^n$  such that  $A(L_1) = L_2$ . This leads to an equivalence relation on  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ , in which two elements of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  are considered to be equivalent if one can be written as the composition of an invertible complex-linear transformation on  $\mathbf{C}^n$  with the other element of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ . In other words, we look at the action of  $GL(\mathbf{C}^n)$  on  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  by post-composition.

Actually, it is more convenient to consider  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ , which we define to be the subset of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  consisting of invertible real-linear transformations from  $\mathbf{R}^{2n}$  to  $\mathbf{C}^n$  such that the image of the first  $n$  standard basis vectors in  $\mathbf{R}^{2n}$  are linearly-independent over the complex numbers as  $n$  vectors in  $\mathbf{C}^n$ . This restriction is not too serious, and indeed we can describe the lattices in  $\mathbf{C}^n$  as images of  $\mathbf{Z}^{2n}$  under mappings in  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ . In other words, if we start with a lattice  $L$  given as the image of  $\mathbf{Z}^{2n}$  under an element of  $\mathcal{L}^*(\mathbf{R}^{2n}, \mathbf{C}^n)$ , we can rewrite it as the image of  $\mathbf{Z}^{2n}$  under a linear transformation in  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  by pre-composing the initial linear transformation from  $\mathbf{R}^{2n}$  to  $\mathbf{C}^n$  with an invertible linear transformation on  $\mathbf{R}^{2n}$  which permutes the standard basis vectors in a suitable way.

To deal with the action of  $GL(\mathbf{C}^n)$  by post-composition, we can restrict ourselves to  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$ , which we define to be the space of invertible real-linear transformations from  $\mathbf{R}^{2n}$  to  $\mathbf{C}^n$  such that the images of the first  $n$  standard basis vectors in  $\mathbf{R}^{2n}$  are the  $n$  standard basis vectors in  $\mathbf{C}^n$ , and in the same order. In other words, if we identify  $\mathbf{R}^{2n}$  with the Cartesian product

$\mathbf{R}^n \times \mathbf{R}^n$ , then these are the invertible real-linear transformations from  $\mathbf{R}^n \times \mathbf{R}^n$  onto  $\mathbf{C}^n$  with the property that on  $\mathbf{R}^n \times \{0\}$  they coincide with the standard embedding of  $\mathbf{R}^n$  into  $\mathbf{C}^n$ . This exactly compensates for the action of  $GL(\mathbf{C}^n)$  by post-composition, since for any collection  $v_1, \dots, v_n$  of linearly-independent vectors in  $\mathbf{C}^n$  there is a unique  $A \in GL(\mathbf{C}^n)$  such that  $A(v_1), \dots, A(v_n)$  are the standard basis vectors in  $\mathbf{C}^n$ , in order.

We can identify  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$  with an open subset of  $\mathcal{L}(\mathbf{R}^n, \mathbf{C}^n)$ . That is, elements of  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$  can be identified with linear transformations from  $\mathbf{R}^n \times \mathbf{R}^n$  into  $\mathbf{C}^n$ , and these linear transformations are determined by what they do on  $\{0\} \times \mathbf{R}^n$ , since their behavior on  $\mathbf{R}^n \times \{0\}$  is fixed by definition. We can think of elements of  $\mathcal{L}(\mathbf{R}^n, \mathbf{C}^n)$  as being written as  $A + iB$ , where  $A, B$  are linear transformations on  $\mathbf{R}^n$ , and one can check that the elements of  $\mathcal{L}^{**}(\mathbf{R}^{2n}, \mathbf{C}^n)$  correspond exactly to elements of  $\mathcal{L}(\mathbf{R}^n, \mathbf{C}^n)$  of the form  $A + iB$ , where  $A, B$  are linear transformations on  $\mathbf{R}^n$  and  $B$  is invertible.

To be more precise, it is helpful to think in terms of real-linear mappings on  $\mathbf{C}^n$ , which can be written as

$$(3.7) \quad T(x + iy) = E_1(x) + E_2(y) + i(E_3(x) + E_4(y)),$$

where  $x, y \in \mathbf{R}^n$ . The passage to  $\mathcal{L}_1^*(\mathbf{R}^{2n}, \mathbf{C}^n)$  can be expressed in these terms as the restriction to invertible real-linear transformations  $T$  on  $\mathbf{C}^n$  of the form

$$(3.8) \quad T(x + iy) = x + A(y) + iB(y),$$

where  $A, B$  are linear transformations on  $\mathbf{R}^n$ . The condition of invertibility of  $T$  is equivalent to the invertibility of  $B$  on  $\mathbf{C}^n$ .

Another way to look at real-linear mappings on  $\mathbf{C}^n$  is as mappings of the form

$$(3.9) \quad T(z) = M(z) + \overline{N(z)},$$

where  $z \in \mathbf{C}^n$ ,  $M$  and  $N$  are complex-linear mappings on  $\mathbf{C}^n$ , and for  $w \in \mathbf{C}^n$ ,  $\overline{w}$  is the element of  $\mathbf{C}^n$  whose coordinates are the complex-conjugates of the coordinates of  $w$ .

Invertibility of  $T$  is a bit tricky, and as an important special case, it is natural to restrict our attention to mappings  $T$  as above for which  $M$  majorizes  $N$  in the sense that

$$(3.10) \quad |N(z)| < |M(z)|$$

for  $z \in \mathbf{C}^n$ ,  $z \neq 0$ , where  $|w|$  denotes the standard Euclidean norm of  $w \in \mathbf{C}^n$ . To factor out the action of  $GL(\mathbf{C}^n)$  by post-composition, we can restrict our attention to real-linear transformations  $T$  of the form

$$(3.11) \quad T(z) = z + \overline{E(z)},$$

where  $E$  is a complex-linear transformation on  $\mathbf{C}^n$  with operator norm strictly less than 1, which is equivalent to saying that  $E^* E < I$ . This has nice features when we think of the image of the standard integer lattice  $(\mathbf{Z}[i])^n$  under  $T$ , with points in the image being reasonably-close to their counterparts in the original lattice.

The  $n = 1$  case is quite instructive. We can write a real-linear transformation  $T$  on  $\mathbf{C}$  as

$$(3.12) \quad T(x + i y) = a x + i b y$$

for  $x, y \in \mathbf{R}$ , where  $a, b$  are complex numbers, and when  $T$  is invertible we can rewrite this as

$$(3.13) \quad T(x + i y) = a(x + i c y),$$

where  $a, c$  are complex numbers with  $a \neq 0$  and  $c$  having nonzero imaginary part. Alternatively, we can write a real-linear transformation  $T$  on  $\mathbf{C}$  as  $T(z) = \alpha z + \beta \bar{z}$  with  $\alpha, \beta \in \mathbf{C}$ , and where  $T$  is invertible if and only if  $|\alpha| \neq |\beta|$ , and when  $|\alpha| > |\beta|$  this can be rewritten as

$$(3.14) \quad T(z) = \theta(z + \mu \bar{z}),$$

where  $\theta$  is a nonzero complex number and  $\mu$  is a complex number such that  $|\mu| < 1$ .

Let us return now to the real case. Consider the quotient space  $O(\mathbf{R}^n) \backslash GL(\mathbf{R}^n)$ , in which two invertible linear transformations on  $\mathbf{R}^n$  are identified if one can be written as an orthogonal linear transformation times the other. We can identify this quotient space with the space of symmetric linear transformations on  $\mathbf{R}^n$  which are positive definite, through the mapping

$$(3.15) \quad T \mapsto T^* T.$$

In other words, if  $T$  is an invertible linear transformation on  $\mathbf{R}^n$ , then  $T^* T$  is a symmetric linear transformation on  $\mathbf{R}^n$  which is positive-definite,  $T_1^* T_1 = T_2^* T_2$  for  $T_1, T_2 \in GL(\mathbf{R}^n)$  if and only if  $T_2 = R T_1$  for some orthogonal transformation  $R$ , and every symmetric linear transformation on  $\mathbf{R}^n$  which is positive-definite can be expressed as  $T^* T$  for an invertible linear transformation  $T$ .

Similarly, the quotient  $SO(\mathbf{R}^n) \backslash SL(\mathbf{R}^n)$  can be identified with the space  $\mathcal{M}(\mathbf{R}^n)$  of symmetric linear transformations on  $\mathbf{R}^n$  which are positive definite and have determinant equal to 1. Let us write  $\Sigma(\mathbf{R}^n)$  for the elements of  $SL(\mathbf{R}^n)$  whose matrices with respect to the standard basis have integer entries. The inverse of a linear transformation in  $\Sigma(\mathbf{R}^n)$  also lies in  $\Sigma(\mathbf{R}^n)$ , because Cramer's rule gives a formula for the matrix of the inverse which shows that it has integer entries when the original matrix has integer entries and determinant equal to 1.

Elements of  $\Sigma(\mathbf{R}^n)$  can be described as the invertible linear transformations which take  $\mathbf{Z}^n$  onto itself. The quotient  $SL(\mathbf{R}^n)/\Sigma(\mathbf{R}^n)$  describes the space of lattices  $L$  in  $\mathbf{R}^n$  such that the corresponding quotient  $\mathbf{R}^n/L$  has volume equal to 1 and for which there is an extra piece of data concerning orientation, and the double quotient  $SO(\mathbf{R}^n) \backslash SL(\mathbf{R}^n)/\Sigma(\mathbf{R}^n)$  deals with these lattices up to equivalence under rotation. By identifying  $SO(\mathbf{R}^n) \backslash SL(\mathbf{R}^n)$  with  $\mathcal{M}(\mathbf{R}^n)$ , the double quotient can be identified with the quotient of  $\mathcal{M}(\mathbf{R}^n)$  by the action of  $\Sigma(\mathbf{R}^n)$  defined by  $A \mapsto T^* A T$ ,  $A \in \mathcal{M}(\mathbf{R}^n)$ ,  $T \in \Sigma(\mathbf{R}^n)$ . This quotient is denoted  $\mathcal{M}(\mathbf{R}^n)/\Sigma(\mathbf{R}^n)$ .

In the complex case let us consider lattices  $L$  in  $\mathbf{C}^n$  which are of the form  $A((\mathbf{Z}[i])^n)$  for some invertible complex-linear mapping  $A$  on  $\mathbf{C}^n$ . It is natural to

look at these lattices up to unitary equivalence, which is to say that two lattices  $L_1, L_2$  are equivalent if there is a unitary linear transformation  $T$  on  $\mathbf{C}^n$  such that  $T(L_1) = L_2$ . This leads to an equivalence relation on  $GL(\mathbf{C}^n)$ , in which two invertible linear transformations  $A_1, A_2$  on  $\mathbf{C}^n$  are considered to be equivalent if there is a unitary linear transformation  $T$  on  $\mathbf{C}^n$  such that  $A_2 = T A_1$ . The quotient of  $GL(\mathbf{C}^n)$  by this equivalence relation is denoted

$$(3.16) \quad U(\mathbf{C}^n) \backslash GL(\mathbf{C}^n)$$

and can be identified with the space of self-adjoint linear transformations on  $\mathbf{C}^n$  which are positive definite, through the mapping

$$(3.17) \quad A \in GL(\mathbf{C}^n) \mapsto A^* A.$$

That is, for each element  $A$  of  $GL(\mathbf{C}^n)$ , the product  $A^* A$  is a self-adjoint linear transformation on  $\mathbf{C}^n$  which is positive-definite,  $A_1^* A_1 = A_2^* A_2$  for two elements  $A_1, A_2$  of  $GL(\mathbf{C}^n)$  if and only if  $A_2 = T A_1$  for some unitary linear transformation  $T$  on  $\mathbf{C}^n$ , and every self-adjoint linear transformation on  $\mathbf{C}^n$  can be expressed as  $A^* A$  for some invertible linear transformation  $A$  on  $\mathbf{C}^n$ .

Similarly, one can consider two elements  $B_1, B_2$  of  $SL(\mathbf{C}^n)$  to be equivalent when there is a linear transformation  $U$  in the special unitary group  $SU(\mathbf{C}^n)$  such that  $A_2 = U A_1$ . The quotient  $SU(\mathbf{C}^n) \backslash SL(\mathbf{C}^n)$  can be identified with the space of self-adjoint linear transformations on  $\mathbf{C}^n$  which are positive-definite and have determinant 1, through the same mapping as before. Let us consider lattices  $L$  of the form  $B((\mathbf{Z}[i])^n)$  for some  $B \in SL(\mathbf{C}^n)$ , a modest normalization.

As in the real case we write  $\Sigma(\mathbf{C}^n)$  for the subgroup of  $SL(\mathbf{C}^n)$  of linear transformations whose associated  $n \times n$  matrices, with respect to the standard basis for  $\mathbf{C}^n$ , have integer entries, which implies that the matrices associated to their inverses also have integer entries. Thus  $B((\mathbf{Z}[i])^n) = (\mathbf{Z}[i])^n$  when  $B \in \Sigma(\mathbf{R}^n)$ , and conversely  $B \in SL(\mathbf{C}^n)$  and  $B((\mathbf{Z}[i])^n) = (\mathbf{Z}[i])^n$  implies that  $B \in \Sigma(\mathbf{C}^n)$ . The quotient  $SL(\mathbf{C}^n)/\Sigma(\mathbf{C}^n)$  represents the space of lattices under consideration, the double quotient  $SU(\mathbf{C}^n) \backslash SL(\mathbf{C}^n)/\Sigma(\mathbf{C}^n)$  represents the space of these lattices modulo equivalence under special unitary transformations, and this double quotient can be identified with the quotient of the space  $\mathcal{M}(\mathbf{C}^n)$  of self-adjoint positive-definite linear transformations on  $\mathbf{C}^n$  with determinant 1 by the action of  $\Sigma(\mathbf{C}^n)$  defined by  $P \mapsto B^* P B$ ,  $B \in \Sigma(\mathbf{C}^n)$ .

Next we consider real and complex projective spaces. Namely, if  $n$  is a positive integer, then the  $n$ -dimensional real and complex projective spaces  $\mathbf{RP}^n, \mathbf{CP}^n$  consist of the real and complex lines through the origin in  $\mathbf{R}^{n+1}, \mathbf{C}^{n+1}$ , respectively. To put it another way, if  $\mathbf{R}_*, \mathbf{C}_*$  denote the nonzero real and complex numbers, respectively, then we have natural actions of  $\mathbf{R}_*, \mathbf{C}_*$  on  $\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{C}^{n+1} \setminus \{0\}$  by scalar multiplication, and the projective spaces are the corresponding quotient spaces. Thus two nonzero vectors  $v, w$  in  $\mathbf{R}^{n+1}, \mathbf{C}^{n+1}$  lead to the same point in the corresponding projective space exactly when they are scalar multiples of each other. Note that we get canonical mappings from  $\mathbf{R}^{n+1} \setminus \{0\}, \mathbf{C}^{n+1} \setminus \{0\}$  onto  $\mathbf{RP}^n, \mathbf{CP}^n$ , in which a nonzero vector  $v$  is sent to the line through the origin which passes through  $v$ .

If  $L$  is a nontrivial linear subspace of  $\mathbf{R}^{n+1}$ ,  $\mathbf{C}^{n+1}$  of dimension  $l + 1$ , say, then we get an interesting space  $\mathbf{P}(L)$  consisting of all lines through the origin in  $L$ , which we can think of as sitting inside of  $\mathbf{RP}^n$ ,  $\mathbf{CP}^n$ , as appropriate. More precisely,  $\mathbf{P}(L)$  is basically a copy of  $\mathbf{RP}^l$  or  $\mathbf{CP}^l$ . These are the  $l$ -dimensional “linear subspaces” of projective space, analogous to linear subspaces of  $\mathbf{R}^n$ ,  $\mathbf{C}^n$ .

If  $A$  is an invertible linear transformation on  $\mathbf{R}^{n+1}$  or on  $\mathbf{C}^{n+1}$ , then  $A$  takes lines to lines, and induces a transformation  $\hat{A}$  on the corresponding projective space. Notice that  $\hat{A}$  is automatically a one-to-one transformation of the corresponding projective space onto itself, with

$$(3.18) \quad \hat{A}^{-1} = (\hat{A^{-1}}),$$

and  $\hat{A}$  maps linear subspaces of projective space to themselves, in the sense of the preceding paragraph. Also, if  $A_1$ ,  $A_2$  are invertible linear transformations on  $\mathbf{R}^{n+1}$  or on  $\mathbf{C}^{n+1}$ , then the induced transformations  $\hat{A}_1$ ,  $\hat{A}_2$  on the corresponding projective space satisfy

$$(3.19) \quad \hat{A}_1 \circ \hat{A}_2 = (\hat{A_1 \circ A_2}).$$

Let  $H$  be a hyperplane in  $\mathbf{R}^{n+1}$  or in  $\mathbf{C}^{n+1}$ , which is to say a linear subspace of dimension  $n$ , and let  $v$  be a nonzero vector in  $\mathbf{R}^{n+1}$ ,  $\mathbf{C}^{n+1}$ , as appropriate. This leads to an affine hyperplane  $H + v$ , consisting of all vectors of the form  $w + v$ ,  $w \in H$ , and which does not contain the vector 0. For each  $w \in H$ , we can look at the line through  $w + v$ , which we can view as an element of the corresponding projective space.

In other words, we basically get an embedding of  $H$  into the corresponding projective space,  $\mathbf{RP}^n$  or  $\mathbf{CP}^n$ . Of course we can also think of  $H$  as being isomorphic to  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , so that we are really looking at a bunch of embeddings of  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  into  $\mathbf{RP}^n$ ,  $\mathbf{CP}^n$ , respectively. For instance, we can do this with  $H$  equal to the  $j$ th coordinate hyperplane in  $\mathbf{R}^{n+1}$ ,  $\mathbf{C}^{n+1}$ ,  $1 \leq j \leq n + 1$ , which is defined by the condition that the  $j$ th coordinate of vectors in  $H$  are equal to 0, and we can take  $v$  to be the  $j$ th standard basis vector, with  $j$ th coordinate equal to 1 and the other  $n$  coordinates equal to 0.

These  $n+1$  embeddings of  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  into  $\mathbf{RP}^n$ ,  $\mathbf{CP}^n$  corresponding to the  $n+1$  coordinate hyperplanes in  $\mathbf{R}^{n+1}$ ,  $\mathbf{C}^{n+1}$  are sufficient to cover the projective space, i.e., every point in projective space shows up in the image of at least one of the embeddings. For a given hyperplane  $H$ , the set of points in the projective space which do not occur in the embedding of  $H$  is the same as  $\mathbf{P}(H)$ . Thus the set of missing points in the projective space lie in a projective subspace of dimension 1 less.

Using these embeddings of  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  into the corresponding projective spaces, we can think of the projective spaces as being manifolds. That is, these embeddings provide local coordinates for all points in the projective space. Two different embeddings which contain the same point  $p$  in their image are compatible in terms of topology and also smooth structure, in the complex case we can say that  $\mathbf{CP}^n$  is a complex manifold. In the real and complex situations there is

a finer “projective” structure which is reflected in the presence of nice projective subspaces, for instance, and the projectivized versions of linear transformations on  $\mathbf{R}^{n+1}$ ,  $\mathbf{C}^{n+1}$ .

Note that two invertible linear transformations  $A_1, A_2$  on  $\mathbf{R}^{n+1}$  or on  $\mathbf{C}^{n+1}$  lead to the same induced transformation on projective space if and only if there is a nonzero scalar  $\alpha$  such that  $A_2 = \alpha A_1$ . Thus the group of these “projective linear transformations” has dimension  $(n+1)^2 - 1$  over the real or complex numbers, as appropriate. Also, for any pair of points  $p, q$  in a projective space, there is a projective linear transformation which takes  $p$  to  $q$ .

If  $k, n$  are positive integers with  $k < n$ , then the Grassmann spaces  $G_{\mathbf{R}}(k, n)$ ,  $G_{\mathbf{C}}(k, n)$  consist of the  $k$ -dimensional linear subspaces of  $\mathbf{R}^n$ ,  $\mathbf{C}^n$ , respectively. When  $k = 1$  this reduces to the  $(n-1)$ -dimensional projective spaces. Suppose that  $L, M$  are linear subspaces of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$  which are complementary, in the sense that

$$(3.20) \quad L \cap M = \{0\} \quad \text{and} \quad L + M = \mathbf{R}^n \text{ or } \mathbf{C}^n,$$

as appropriate, and that  $L$  has dimension  $k$ , so that  $M$  has dimension  $n - k$ . If  $A$  is a linear mapping from  $L$  to  $M$ , then the graph of  $A$ , consisting of the vectors

$$(3.21) \quad v + A(v), \quad v \in L,$$

is also a  $k$ -dimensional subspace of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$ , as appropriate. In this way we can embed the vector space of linear transformations from  $L$  to  $M$  into the Grassmannian, and this provides a nice coordinate patch around  $L$  itself.

In particular, these coordinate patches permit one to view the Grassmann spaces as smooth manifolds, and as complex manifolds in the complex case. The dimension of  $G_{\mathbf{R}}(k, n)$ ,  $G_{\mathbf{C}}(k, n)$  is equal to

$$(3.22) \quad k(n - k)$$

with respect to the real or complex numbers, as appropriate. Just as for projective spaces, invertible linear transformations on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  induce interesting mappings on the corresponding Grassmannians. These actions are again transitive, because if  $L_1, L_2$  are  $k$ -dimensional linear subspaces of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$ , then there is an invertible linear transformation  $A$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ , as appropriate, such that  $A(L_1) = L_2$ .

There is a natural correspondence between the Grassmann spaces of  $k$ -dimensional linear subspaces in  $\mathbf{R}^n$  or in  $\mathbf{C}^n$  and the Grassmann spaces of  $(n - k)$ -dimensional linear subspaces of  $\mathbf{R}^n$  or in  $\mathbf{C}^n$ , respectively. More precisely, there is a natural correspondence between  $k$ -dimensional linear subspaces of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$  and  $(n - k)$ -dimensional linear subspaces of the dual spaces associated to  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , consisting of the linear functionals on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ . Namely, if  $L$  is a  $k$ -dimensional linear subspace of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$ , then one gets an  $(n - k)$ -dimensional linear subspace of the dual space by taking the linear functionals which vanish on  $L$ . Conversely, if one starts with an  $(n - k)$ -dimensional subspace of the dual space, one gets a  $k$ -dimensional subspace of the original space by taking the intersections of the kernels of the linear functionals in the

subspace of the dual space. Using the standard basis for  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , or any other basis for that matter, one can identify  $\mathbf{R}^n$ ,  $\mathbf{C}^n$  with their dual spaces in a well-known manner.

Suppose that  $L$  is a linear subspace of  $\mathbf{R}^n$  or of  $\mathbf{C}^n$  of dimension  $l$ . If  $l \geq k$ , then we get an interesting subspace of the Grassmann space of  $k$ -dimensional linear subspaces of  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, consisting of the  $k$ -dimensional linear subspaces contained in  $L$ . If  $k \geq l$ , then there is another interesting subspace of the Grassmann space, consisting of the  $k$ -dimensional linear subspaces of  $\mathbf{R}^n$  or  $\mathbf{C}^n$  which contain  $L$ . When  $k = l$ , these two cases are the same and we get simply a point in the Grassmann space.

Let  $n$  be a positive integer. By a *multiindex* we mean an  $n$ -tuple  $\alpha = (\alpha_1, \dots, \alpha_n)$  of nonnegative integers, and in this case we set

$$(3.23) \quad |\alpha| = \sum_{j=1}^n |\alpha_j|.$$

Given a multiindex  $\alpha$ , we define the corresponding monomial  $w^\alpha$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  by

$$(3.24) \quad w^\alpha = w_1^{\alpha_1} \cdots w_n^{\alpha_n},$$

where  $w = (w_1, \dots, w_n)$  as usual, and we call  $|\alpha|$  the *degree* of this monomial. When  $\alpha_j = 0$  we interpret  $w_j^{\alpha_j}$  as being equal to the constant 1.

A polynomial  $p(w)$  on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  is a function which is a linear combination of monomials. We take the coefficients to be real or complex numbers according to whether we are working on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$ . A polynomial  $p(w)$  which is a linear combination of monomials of the same degree  $a$  is said to be homogeneous of degree  $a$ . This is equivalent to the condition that

$$(3.25) \quad p(\lambda w) = \lambda^a p(w)$$

for all real or complex numbers  $\lambda$  and all  $w$  in  $\mathbf{R}^n$  or in  $\mathbf{C}^n$ , as appropriate.

Suppose that  $p_1(w), \dots, p_{n+1}(w)$  are polynomials on  $\mathbf{R}^{n+1}$  or on  $\mathbf{C}^{n+1}$  which are homogeneous of the same degree  $a > 0$ . Assume also that

$$(3.26) \quad p_1(w) = \cdots = p_{n+1}(w) = 0$$

only when  $w = 0$ . The combined mapping  $p(w) = (p_1(w), \dots, p_{n+1}(w))$  is a homogeneous polynomial mapping of  $\mathbf{R}^{n+1}$  or  $\mathbf{C}^{n+1}$  into itself which maps nonzero vectors to nonzero vectors, and as a result induces a mapping  $\widehat{p}$  from  $\mathbf{RP}^n$  or  $\mathbf{CP}^n$  to itself, as appropriate. The degree 1 case corresponds exactly to invertible linear mappings on  $\mathbf{R}^{n+1}$  or on  $\mathbf{C}^{n+1}$  and the associated projective linear transformations on the corresponding projective spaces. If  $p = (p_1, \dots, p_{n+1})$ ,  $q = (q_1, \dots, q_{n+1})$  are homogeneous polynomial mappings on  $\mathbf{R}^{n+1}$  or on  $\mathbf{C}^{n+1}$  of this type, of degrees  $a, b > 0$ , respectively, then the composition  $p \circ q$  is a homogeneous polynomial mapping of degree  $ab$ , and  $\widehat{p \circ q} = \widehat{p} \circ \widehat{q}$ .

When  $n = 1$ , we can think of  $\mathbf{RP}^1$ ,  $\mathbf{CP}^1$  as being the same as  $\mathbf{R}$ ,  $\mathbf{C}$  with an additional point added, often denoted  $\infty$ . Projective linear transformations on



$\mathbf{RP}^1$ ,  $\mathbf{CP}^1$  can then be described as mappings of the form  $(az + b)/(cz + d)$ , where  $ad - bc \neq 0$ , and using standard conventions along the lines of  $1/0 = \infty$ ,  $1/\infty = 0$ . Similarly, the mappings associated to homogeneous polynomials as in the preceding paragraph reduce to nonconstant rational functions of a single variable.

## 4 Immersions, submersions, and connections

Let  $M$ ,  $N$  be nonempty  $m$ ,  $n$ -dimensional smooth manifolds, so that  $M$ ,  $N$  look locally like  $\mathbf{R}^m$ ,  $\mathbf{R}^n$  in a sense, and they have countable bases for their topologies. As basic situations,  $M$ ,  $N$  might in fact be open subsets of  $\mathbf{R}^m$ ,  $\mathbf{R}^n$ , respectively. They might instead be given as embedded submanifolds of higher-dimensional Euclidean spaces.

Suppose that  $f$  is a smooth mapping from  $M$  into  $N$ . For each point  $p \in M$ , the differential of  $f$  at  $p$  is denoted  $df_p$  and is a linear mapping from the tangent space of  $M$  at  $p$ , which is denoted  $T_p M$  and is an  $m$ -dimensional real vector space, into the tangent space  $T_{f(p)} N$  of  $N$  at  $f(p)$ , an  $n$ -dimensional real vector space. We say that  $f$  is an *immersion* if  $df_p$  is an injective linear transformation from  $T_p M$  into  $T_{f(p)} N$  for each  $p \in M$ , which is to say that the kernel of  $df_p$  is trivial for all  $p \in M$ , and in which case  $m \leq n$ . We say that  $f$  is a *submersion* if  $df_p$  maps  $T_p M$  onto  $T_{f(p)} N$  for all  $p \in M$ , in which case  $m \geq n$ . When  $m = n$ , these two conditions are equivalent to each other, and to the statement that  $df_p$  is a one-to-one linear mapping from  $T_p M$  onto  $T_{f(p)} N$  for each  $p \in M$ .

When  $m \leq n$ , we have the standard embedding of  $\mathbf{R}^m$  into  $\mathbf{R}^n$ , in which a point  $x = (x_1, \dots, x_m)$  in  $\mathbf{R}^m$  is sent to  $\hat{x} = (\hat{x}_1, \dots, \hat{x}_n)$  in  $\mathbf{R}^n$ , with  $\hat{x}_i = x_i$  for  $1 \leq i \leq m$  and  $\hat{x}_i = 0$  for  $m < i \leq n$ . When  $m \geq n$ , we have the standard projection from  $\mathbf{R}^m$  onto  $\mathbf{R}^n$ , in which one keeps the first  $n$  coordinates of a point in  $\mathbf{R}^m$  and drops the remaining  $m - n$  coordinates. By the implicit function theorem, immersions and submersions are locally equivalent to these standard models, with the appropriate dimensions. When  $m = n$ , immersions and submersions are the same, and they are local diffeomorphisms, as in the inverse function theorem.

If  $f : M \rightarrow N$  is a submersion, then  $f$  is an open mapping in particular, which is to say that  $f(U)$  is an open subset of  $N$  whenever  $U$  is an open subset of  $M$ . If we also assume that  $f$  is proper, in the sense that  $f^{-1}(K)$  is a compact subset of  $M$  whenever  $K$  is a compact subset of  $N$ , then it is easy to check that  $f(E)$  is a closed subset of  $N$  when  $E$  is a closed subset of  $M$ . This implies in turn that  $f(M) = N$  when  $N$  is connected, since  $f(M)$  would then be a nonempty subset of  $N$  which is both open and closed.

Let us consider some examples. Fix a positive integer  $n$ , and for  $M$  take  $\mathbf{R}^{n+1} \setminus \{0\}$ . For  $N$  we can take the projective space  $\mathbf{RP}^n$ , and we have a natural mapping from  $\mathbf{R}^{n+1} \setminus \{0\}$  to  $\mathbf{RP}^n$  which sends a nonzero vector  $v$  in  $\mathbf{R}^{n+1}$  to the point in  $\mathbf{RP}^n$  corresponding to the line through  $v$ . This mapping is smooth and defines a submersion.

We can also restrict this mapping to the unit sphere  $\mathbf{S}^n$  in  $\mathbf{R}^{n+1}$ , consisting

of the vectors  $v$  such that  $|v| = 1$ . This mapping is still a smooth mapping from  $\mathbf{S}^n$  onto  $\mathbf{RP}^n$ , with the dimensions of the domain and range now being the same. This mapping is a local diffeomorphism, and of course two vectors  $v, w$  in  $\mathbf{S}^n$  are sent to the same point in  $\mathbf{RP}^n$  if and only if  $v = w$  or  $v = -w$ .

Now take  $M$  to be  $\mathbf{C}^{n+1} \setminus \{0\}$ , which has real dimension  $2n + 2$ , and let  $N$  be  $\mathbf{CP}^n$ , which has real dimension  $2n$ . Once again there is a natural mapping which sends a nonzero vector  $v$  in  $\mathbf{C}^{n+1}$  to the point in complex projective space  $\mathbf{CP}^n$  that corresponds to the line through  $v$ . This mapping is smooth, and in fact holomorphic, and it is also a submersion. We can also restrict this mapping to the sphere  $\mathbf{S}^{2n+1}$  consisting of the vectors  $v$  in  $\mathbf{C}^{n+1}$  such that  $|v| = 1$ , to get a submersion onto  $\mathbf{CP}^n$ . Two vectors  $v, w$  in  $\mathbf{S}^{2n+1}$  are sent to the same point in  $\mathbf{CP}^n$  by this mapping if and only if  $w = \alpha v$  for some complex number  $\alpha$  such that  $|\alpha| = 1$ , so that the fibers of this submersion from  $\mathbf{S}^{2n+1}$  onto  $\mathbf{CP}^n$  are all circles.

Suppose that  $M, N$  are smooth manifolds of dimensions  $m, n$ , respectively, and that  $f : M \rightarrow N$  is a proper smooth submersion, so that the fibers  $f^{-1}(z)$ ,  $z \in N$ , are compact submanifolds of  $M$  of dimension  $m - n$ . Fix a point  $z_1$  in  $N$ , and suppose that  $V_1$  is a neighborhood of  $z_1$  in  $N$  and that  $\phi$  is a smooth mapping from  $f^{-1}(V_1) \subseteq M$  into  $f^{-1}(z_1)$  such that  $\phi(x) = x$  when  $f(x) = z_1$ . We can combine  $f, \phi$  to get a smooth mapping  $(f, \phi)$  from  $f^{-1}(V_1)$  into  $V_1 \times f^{-1}(z_1)$ . The differential of this combined mapping is invertible at each element of the fiber  $f^{-1}(z_1)$ , and it follows that there is a neighborhood  $V_2$  of  $z_1$  contained in  $V_1$  such that the combined mapping  $(f, \phi)$  defines a diffeomorphism from  $f^{-1}(V_2)$  onto  $V_2 \times f^{-1}(z_1)$ . In particular, if  $z_2$  is an element of  $N$  which is sufficiently close to  $z_1$ , then the fibers  $f^{-1}(z_1), f^{-1}(z_2)$  are diffeomorphic smooth manifolds of dimension  $m - n$ . If  $N$  is connected, then it follows that all of the fibers  $f^{-1}(z), z \in N$ , are diffeomorphic to each other.

Again let  $M, N$  be smooth manifolds of dimensions  $m, n$ , respectively, and let  $f : M \rightarrow N$  be a smooth submersion which may or may not be proper, at least for the moment. For each  $p \in M$ , we get a linear subspace  $V_p$  of the tangent space  $T_p M$  consisting of the tangent vectors to  $M$  at  $p$  which are also tangent to the fiber  $f^{-1}(f(p))$  of  $f$  passing through  $p$ . This can also be described as the kernel of the differential  $df_p$  of  $f$  at  $p$ , as a linear mapping from  $T_p M$  to  $T_{f(p)} N$ . Of course  $V_p$  has dimension  $m - n$  for each  $p \in M$ , because  $f$  is a submersion.

By a *connection* on  $M$  with respect to the submersion  $f : M \rightarrow N$  we mean a choice of an  $n$ -dimensional linear subspace  $H_p$  of the tangent space  $T_p M$  for each point  $p \in M$  which is transverse to the vertical subspace  $V_p$  and which depends smoothly on  $p$ . We call  $H_p$  the horizontal linear subspace of the tangent space  $T_p M$  determined by the connection, and the restriction of the differential  $df_p$  of  $f$  at  $p$  to the horizontal subspace  $H_p$  of  $T_p M$  is a one-to-one linear mapping of  $H_p$  onto the tangent space  $T_{f(p)} N$  of  $N$  at  $f(p)$ . To put it another way, at each point  $p \in M$  the tangent space  $T_p M$  of  $M$  at  $p$  is the direct sum of the horizontal and vertical subspaces  $H_p$  and  $V_p$ . The vertical subspace  $V_p$  is determined by  $f$ , and there is some room for making choices for the horizontal subspaces.

As a basic scenario, suppose that  $M$  is also equipped with a smooth Riemannian metric, which is to say an inner product on each tangent space  $T_p M$

which depends smoothly on  $p$ . In this case, one can simply choose  $H_p$  to be the orthogonal complement of  $V_p$  in  $T_p M$  with respect to the Riemannian metric. This shows that connections always exist, since Riemannian metrics exist on any smooth manifold. Recall that one way to choose a Riemannian metric on a smooth manifold  $M$  is to choose local Riemannian metrics in coordinate charts and combine them using a partition of unity, and another way is to embed  $M$  into a Euclidean space and then use the Riemannian metric inherited from the one on the Euclidean space.

Let  $A$  be an invertible linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  such that

$$(4.1) \quad \lim_{l \rightarrow \infty} A^l(v) = 0$$

for all  $v$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate. A sufficient condition for this to hold is that the norm of  $A$  be strictly less than 1. In the complex case, this condition holds if and only if the eigenvalues of  $A$  all have modulus strictly less than 1, and in the real case one can complexify  $\mathbf{R}^n$  to convert  $A$  to a linear transformation on  $\mathbf{C}^n$ , and the condition holds on  $\mathbf{R}^n$  if and only if it holds for the complexification on  $\mathbf{C}^n$ , which is to say that the modulus of each of the eigenvalues of the associated linear transformation on  $\mathbf{C}^n$  should be strictly less than 1. Notice that our condition is also equivalent to

$$(4.2) \quad \lim_{l \rightarrow \infty} |A^{-l}(v)| = \infty$$

for all nonzero vectors  $v$  in  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate.

Let us define a space  $\mathcal{H}_A$  by starting with  $\mathbf{R}^n \setminus \{0\}$  or  $\mathbf{C}^n \setminus \{0\}$ , as appropriate, and identifying two nonzero vectors  $v, w$  when

$$(4.3) \quad w = A^j(v)$$

for some integer  $j$ . Thus  $\mathcal{H}_A$  is a compact real or complex manifold, according to whether one starts with  $\mathbf{R}^n$  or  $\mathbf{C}^n$ . There is a natural smooth mapping from  $\mathbf{R}^n \setminus \{0\}$  or  $\mathbf{C}^n \setminus \{0\}$  onto  $\mathcal{H}_A$ , as appropriate, in which a nonzero vector  $v$  is mapped to the corresponding equivalence class in  $\mathcal{H}_A$ , and this mapping is holomorphic in the complex case. Of course  $\mathcal{H}_A$  has the same dimension as  $\mathbf{R}^n \setminus \{0\}$  or  $\mathbf{C}^n \setminus \{0\}$ , as appropriate, and the mapping to  $\mathcal{H}_A$  is a local diffeomorphism.

If  $n = 1$ , then  $A(v) = \alpha v$  for some nonzero scalar  $\alpha$  with  $|\alpha| < 1$ . In the real case, if  $\alpha > 0$ , then  $A$  sends the positive real numbers to the positive real numbers and  $A$  sends the negative real numbers to the negative real numbers, and  $\mathcal{H}_A$  is basically a disjoint union of two circles. If  $\alpha < 0$ , then  $A$  maps the positive real numbers to the negative real numbers and vice-versa, and  $\mathcal{H}_A$  reduces to a single circle. In the complex case, we can think of  $\mathbf{C} \setminus \{0\}$  as the same as  $\mathbf{C}/2\pi i\mathbf{Z}$ , i.e., the quotient of  $\mathbf{C}$  by translations by integer multiples of  $2\pi i$ , using the exponential mapping from  $\mathbf{C}$  onto  $\mathbf{C} \setminus \{0\}$ . If  $\beta$  is a complex number such that  $\exp \beta = \alpha$  and  $L$  is the lattice in  $\mathbf{C}$  consisting of complex numbers of the form  $2m\pi i + n\beta$ ,  $m, n \in \mathbf{Z}$ , then  $\mathcal{H}_A$  can be identified with  $\mathbf{C}/L$ , which is to say that we get a 1-dimensional complex torus.

When  $n \geq 2$ , consider the special case where  $A$  is of the form  $A(v) = \alpha v$  for some nonzero scalar  $\alpha$  such that  $|\alpha| < 1$ . In this case there is a natural smooth mapping from  $\mathcal{H}_A$  onto  $\mathbf{RP}^{n-1}$  or  $\mathbf{CP}^{n-1}$ , as appropriate, in which elements of  $\mathcal{H}_A$  are sent to the lines in  $\mathbf{R}^n$  or  $\mathbf{C}^n$  that contain the corresponding vectors. This mapping is a submersion, and it is holomorphic in the complex case. The fibers of this mapping are copies of what one gets in the 1-dimensional case. Also, linear mappings on  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, induce interesting mappings on  $\mathcal{H}_A$ .

In general dimensions and for general  $A$ , suppose that  $L$  is a nontrivial linear subspace of  $\mathbf{R}^n$  or  $\mathbf{C}^n$ , as appropriate, such that

$$(4.4) \quad A(L) = L.$$

We can apply the same construction to get a space analogous to  $\mathcal{H}_A$  for the restriction of  $A$  to  $L$ , and this space can be viewed as a submanifold of  $\mathcal{H}_A$ . In particular, a 1-dimensional invariant subspace for  $A$  is the same as the span of a nonzero eigenvector of  $A$ , and this leads to a submanifold of  $\mathcal{H}_A$  which is a copy of what one gets in the 1-dimensional case. Also, a linear transformation on  $\mathbf{R}^n$  or on  $\mathbf{C}^n$  which commutes with  $A$  leads to an interesting mapping on  $\mathcal{H}_A$ .

Suppose again that  $M, N$  are smooth manifolds of dimensions  $m, n$ , that  $f : M \rightarrow N$  is a submersion, and that we have a connection on  $M$  associated to this submersion, defined by a smooth family  $H_p$  of horizontal linear subspaces of  $T_p M$ ,  $p \in M$ . Fix a point  $p \in M$ , and let  $\alpha(t)$  be a smooth curve in  $N$  defined on an interval  $[a, b]$  in the real line such that

$$(4.5) \quad f(p) = \alpha(a).$$

Consider the question of having a smooth curve  $\beta(t)$ ,  $a \leq t \leq b$ , in  $N$  such that

$$(4.6) \quad \beta(a) = p$$

and

$$(4.7) \quad f(\beta(t)) = \alpha(t) \quad \text{for all } a \leq t \leq b,$$

so that  $\beta(t)$  is a lifting of  $\alpha(t)$  which starts at  $p$ , and also

$$(4.8) \quad \dot{\beta}(t) \in H_{\beta(t)} \quad \text{for all } a \leq t \leq b.$$

Here  $\dot{\beta}(t)$  denotes the derivative of  $\beta(t)$  at  $t$ , which is automatically an element of  $T_{\beta(t)}M$ , and which satisfies

$$(4.9) \quad df_{\beta(t)}(\dot{\beta}(t)) = \dot{\alpha}(t)$$

for all  $t$ . This condition and the requirement that the derivative of  $\beta(t)$  belong to the horizontal subspaces specified by the connection determine the derivative of  $\beta(t)$  in terms of  $\beta(t)$  and the derivative of  $\alpha(t)$ .

In other words,  $\beta(t)$  satisfies an ordinary differential equation. More precisely, in local coordinates one can convert this into a system of ordinary differential equations in  $\mathbf{R}^m$  of the usual type. Standard results about ordinary differential equations imply that  $\beta(t)$  is uniquely determined by  $\alpha(t)$  and the starting point  $p$ , when such a lifting exists. Also, such a lifting always exists at least on a shorter interval beginning at  $a$ . If the submersion  $f : M \rightarrow N$  is proper, then the lifting  $\beta(t)$  exists for the whole interval  $[a, b]$ , and there are other conditions like this for lifting the whole curve as well, basically by ensuring that the lifted curve remain in a compact subset of  $M$ .

Standard results about ordinary differential equations also imply smoothness results about mappings associated to liftings like these. Namely, one can vary the choice of  $p$  in the fiber  $f^{-1}(\alpha(a))$ , and get smooth dependence on  $p$  of the lifting. One can always do this locally, for  $t$  near  $a$ . If  $f : M \rightarrow N$  is proper, so that we have liftings on the whole interval  $[a, b]$ , then the lifting of paths defines a mapping from the fiber  $f^{-1}(\alpha(a))$  to the fiber  $f^{-1}(\alpha(b))$ , and this mapping is a bijection with the inverse mapping obtained by running the lifting backwards along the interval  $[a, b]$ . Smooth dependence on  $p$  implies that this mapping from the fiber  $f^{-1}(\alpha(a))$  onto  $f^{-1}(\alpha(b))$  is in fact a diffeomorphism.

If  $f : M \rightarrow N$  is proper and  $N$  is connected, then we get another way to see that the fibers of  $f$  are all diffeomorphic to each other. Namely, any pair of points in  $N$  can be connected by a smooth curve if  $N$  is connected. One can use liftings of this curve to get a diffeomorphism between the corresponding fibers, as above. Of course there are plenty of variations of these themes.

Let us consider another example. Let  $U$  denote the upper half-space in  $\mathbf{C}$ , which consists of complex numbers with positive imaginary part. We start basically with the Cartesian product  $\mathbf{C} \times U$ , and the coordinate projection of this space onto  $U$ . This projection is obviously a holomorphic mapping and a submersion.

For each  $\alpha \in U$ , let  $L_\alpha$  be the lattice in  $\mathbf{C}$  consisting of  $m + n\alpha$ ,  $m, n \in \mathbf{Z}$ . We can think of this as a fixed lattice in  $\mathbf{C}$ , or as a family of lattices in a family of copies of  $\mathbf{C}$ . For each  $\alpha \in U$ , let us write  $\mathcal{E}(\alpha)$  for the 1-dimensional complex torus  $\mathbf{C}/L_\alpha$ . Let us write  $\mathcal{E}$  for the space that we get by identifying  $(z, \alpha)$  and  $(w, \alpha)$  in  $\mathbf{C} \times U$  when  $w - z \in L_\alpha$ .

Thus  $\mathcal{E}$  is a complex manifold of dimension 2, and we get a holomorphic projection mapping from  $\mathbf{C} \times U$  onto  $\mathcal{E}$  sending a given pair  $(z, \alpha)$  in  $\mathbf{C} \times U$  to the corresponding equivalence class in  $\mathcal{E}$ . Locally  $\mathcal{E}$  looks like  $\mathbf{C} \times U$ , which is to say that the natural quotient mapping is locally a biholomorphism. We also have a natural mapping from  $\mathcal{E}$  onto  $U$ , which is to say that the standard coordinate projection from  $\mathbf{C} \times U$  pushes down to  $\mathcal{E}$  in a natural way. That is, the standard coordinate projection from  $\mathbf{C} \times U$  is the same as the natural projection from  $\mathbf{C} \times U$  onto  $\mathcal{E}$  followed by the mapping from  $\mathcal{E}$  to  $U$ . This mapping from  $\mathcal{E}$  to  $U$  is a proper holomorphic submersion.

For each  $\alpha \in U$ , we can identify the fiber in  $\mathcal{E}$  over  $\alpha$  from our mapping  $\mathcal{E} \rightarrow U$  with  $\mathcal{E}(\alpha)$  in a simple way. As real manifolds, these fibers are all diffeomorphic to each other. However, these fibers are not equivalent in general as complex manifolds. In fact, for distinct nearby  $\alpha$ 's the corresponding  $\mathcal{E}(\alpha)$ 's

are not holomorphically equivalent. Locally  $\mathcal{E}$  is holomorphically equivalent to a product, and there is significant activity more globally.

Now let  $M, N$  be smooth manifolds with dimensions  $m, n$ , let  $f : M \rightarrow N$  be a smooth submersion, and let  $H_p, p \in M$ , be a smooth family of horizontal linear subspaces of the tangent spaces of  $M$  defining a connection for the submersion. Thus each  $H_p$  has dimension  $n$  and the restriction of the differential  $df_p$  of  $f$  at  $p$  to  $H_p$  defines a one-to-one linear mapping onto  $T_{f(p)}N$ . We would like to describe the *curvature* of this connection. Let us first review some aspects of *vector fields* on a smooth manifold.

Let  $U$  be a nonempty open subset of  $M$ . A smooth vector field  $X$  on  $U$  assigns to each  $p \in U$  a tangent vector  $X(p)$  to  $M$  at  $p$ , in a way which is smooth in  $p$ . Such a vector field defines a first-order linear differential operator acting on smooth real-valued functions on  $U$ , which is to say that if  $h$  is a smooth real-valued function on  $U$ , then  $X(h)$  at a point  $p \in U$  is the directional derivative of  $h$  in the direction  $X(p)$  at  $p$ . These differential operators are linear, so that

$$(4.10) \quad X(c_1 h_1 + c_2 h_2) = c_1 X(h_1) + c_2 X(h_2)$$

for all real numbers  $c_1, c_2$  and all smooth functions  $h_1, h_2$ , and they satisfy the Leibniz rule

$$(4.11) \quad X(h_1 h_2) = X(h_1) h_2 + h_1 X(h_2),$$

for differentiating the product of two smooth functions  $h_1, h_2$  on  $U$ .

If  $X_1, X_2$  are two smooth vector fields on  $U$ , then one gets the associated Lie bracket  $[X_1, X_2]$  of  $X_1, X_2$ . In terms of differential operators, we have

$$(4.12) \quad [X_1, X_2](h) = X_1(X_2(h)) - X_2(X_1(h))$$

for all smooth functions  $h$  on  $U$ . Of course  $X_1(X_2(h)), X_2(X_1(h))$  involve second derivatives of  $h$ , and these second derivatives cancel out in the difference, leaving a first-order operator associated to a vector field. If  $X_1, X_2$  are smooth vector fields on  $U$  and  $\phi_1, \phi_2$  are smooth real-valued functions on  $U$ , then  $\phi_1 X_1, \phi_2 X_2$  also define smooth vector fields on  $U$ , and we have that

$$(4.13) \quad [\phi_1 X_1, \phi_2 X_2] = \phi_1 \phi_2 [X_1, X_2] + \phi_1 X_1(\phi_2) X_2 - \phi_2 X_2(\phi_1) X_1.$$

Now let us return to the setting of our submersion and connection, and assume that  $X_1, X_2$  are smooth vector fields on a nonempty open subset  $U$  of  $M$  such that  $X_1(p), X_2(p)$  are elements of the horizontal linear subspace  $H_p$  of the tangent space  $T_p M$  of  $M$  at  $p$  for each  $p \in U$ . Define the curvature  $\mathcal{C}(X_1, X_2)$  at a point  $p \in U$  to be the vertical component of  $[X_1, X_2]$  at  $p$ , which is to say that  $\mathcal{C}(X_1, X_2)$  at  $p$  is an element of the vertical linear subspace  $V_p$  of the tangent space  $T_p M$  of  $M$  at  $p$ , and  $[X_1, X_2] - \mathcal{C}(X_1, X_2)$  lies in the horizontal subspace  $H_p$  at  $p$ . If  $\phi_1, \phi_2$  are smooth real-valued functions on  $U$ , then  $\phi_1 X_1, \phi_2 X_2$  are also horizontal vector fields on  $U$ , and

$$(4.14) \quad \mathcal{C}(\phi_1 X_1, \phi_2 X_2) = \phi_1 \phi_2 \mathcal{C}(X_1, X_2).$$

This shows that the curvature  $\mathcal{C}(X_1, X_2)$ , at a point  $p \in U$ , depends only on the values of  $X_1, X_2$  at  $p$ , and thus at  $p$  the curvature  $\mathcal{C}(X_1, X_2)$  defines an antisymmetric bilinear mapping from  $H_p \times H_p$  into  $V_p$ , which depends smoothly on  $p$ . Using the isomorphism between  $H_p$  and  $T_{f(p)}N$  given by  $df_p$ , one can reformulate the curvature by saying that for each  $p \in M$  it is an antisymmetric bilinear mapping from  $T_{f(p)}N \times T_{f(p)}N$  into  $V_p$  which depends smoothly on  $p$ .

What does it mean for the curvature to be equal to 0 everywhere on  $M$ ? This is equivalent to saying that if  $X_1, X_2$  are smooth horizontal vector fields on a nonempty subset  $U$  of  $M$ , then the Lie bracket  $[X_1, X_2]$  is also a horizontal vector field on  $U$ . In other words, the curvature of the connection is equal to 0 on all of  $M$  if and only if the corresponding distribution of horizontal linear subspaces of the tangent spaces is integrable. By a well-known theorem of Frobenius, this means that there is a foliation of  $M$  by  $n$ -dimensional smooth submanifolds whose tangent spaces are exactly the horizontal linear subspaces of the tangent spaces of  $M$  given by the connection.

## 5 Metric spaces

By a *metric space* we mean a nonempty set  $M$  together with a real-valued function  $d(x, y)$  defined for  $x, y \in M$ , called the *distance function* or *metric* on  $M$ , such that  $d(x, y) \geq 0$  for all  $x, y \in M$ ,  $d(x, y) = 0$  if and only if  $x = y$ ,

$$(5.1) \quad d(y, x) = d(x, y)$$

for all  $x, y \in M$ , and

$$(5.2) \quad d(x, z) \leq d(x, y) + d(y, z)$$

for all  $x, y, z \in M$ . A basic example is given by the real numbers  $\mathbf{R}$  equipped with the standard metric  $|x - y|$ . Recall that if  $x$  is a real number, then the *absolute value* of  $x$  is denoted  $|x|$  and defined to be equal to  $x$  when  $x \geq 0$  and to  $-x$  when  $x \leq 0$ , and that

$$(5.3) \quad |x + y| \leq |x| + |y|, \quad |xy| = |x||y|$$

for all  $x, y \in \mathbf{R}$ .

If  $(M, d(x, y))$  and  $(N, \rho(u, v))$  are metric spaces and  $f : M \rightarrow N$  is a mapping from  $M$  to  $N$ , then  $f$  is said to be *continuous* if for every  $x \in M$  and every positive real number  $\epsilon$  there is a positive real number  $\delta$  such that

$$(5.4) \quad \rho(f(y), f(x)) < \epsilon$$

for all  $y \in M$  such that  $d(y, x) < \delta$ . For instance, constant mappings are always continuous, and the identity mapping on a metric space  $(M, d(x, y))$  is continuous as a mapping from  $M$  to itself. More generally, if  $(M, d(x, y))$  is a metric space and  $E$  is a nonempty subset of  $M$ , then we can consider  $E$  to be a metric space itself, using the same metric  $d(x, y)$  restricted to  $E$ , and then the inclusion mapping of  $E$  into  $M$ , which sends each element of  $E$  to itself, is continuous as a mapping from  $E$  to  $M$ .

Let  $(M, d(x, y))$  be a metric space, and let  $p$  be a point in  $M$ . From the triangle inequality it is easy to see that

$$(5.5) \quad d(x, p) - d(y, p) \leq d(x, y)$$

for all  $x, y \in M$ , and similarly

$$(5.6) \quad d(y, p) - d(x, p) \leq d(x, y),$$

so that

$$(5.7) \quad |d(x, p) - d(y, p)| \leq d(x, y)$$

for all  $x, y \in M$ . It follows that the real-valued function  $f_p(x) = d(x, p)$  on  $M$  is continuous, where, as usual, we employ the standard metric on the real numbers.

If  $f_1, f_2$  are two real-valued continuous functions on a metric space  $(M, d(x, y))$ , then the sum  $f_1 + f_2$  and the product  $f_1 f_2$  are also continuous functions. This is not too difficult to check. Similarly, if  $f$  is a continuous real-valued function on  $M$  such that  $f(x) \neq 0$  for all  $x \in M$ , then  $1/f(x)$  is also a continuous function on  $M$ .

Let  $(M, d(x, y))$  be a metric space, and let  $A$  be a nonempty subset of  $M$ . We denote the distance from a point  $x$  in  $M$  to  $A$  by  $\text{dist}(x, A)$ , and we define it to be the infimum of  $d(x, y)$  over all  $y \in A$ . If  $A, B$  are two nonempty subsets of  $M$ , then the distance between  $A$  and  $B$  is denoted  $\text{dist}(A, B)$  and defined to be the infimum of  $d(x, y)$  over all  $x \in A$  and  $y \in B$ .

If  $A$  is a nonempty subset of  $M$  and  $x, y$  are elements of  $M$ , then one can check that

$$(5.8) \quad \text{dist}(x, A) \leq \text{dist}(y, A) + d(x, y).$$

As a result,

$$(5.9) \quad |\text{dist}(x, A) - \text{dist}(y, A)| \leq d(x, y)$$

for all  $x, y \in M$ . In particular,  $\text{dist}(x, A)$  is a real-valued continuous function of  $x$  on  $M$ .

If  $A$  and  $B$  are nonempty subsets of  $M$  and  $t$  is a positive real number, then we say that  $A, B$  are  $t$ -close if for each  $a \in A$  there is a  $b \in B$  such that  $d(a, b) < t$ , and if for each  $b \in B$  there is an  $a \in A$  such that  $d(a, b) < t$ . If  $A, B$  are nonempty subsets of  $M$  which are  $t$ -close for some positive real number  $t$ , then the Hausdorff distance from  $A$  to  $B$  is denoted  $D(A, B)$  and defined to be the infimum of the positive real numbers  $t$  such that  $A, B$  are  $t$ -close. In this case, if  $x$  is any point in  $M$ , then

$$(5.10) \quad \text{dist}(x, A) \leq \text{dist}(x, B) + D(A, B).$$

A subset  $A$  of  $M$  is said to be *bounded* if there is a point  $p$  in  $M$  and a positive real number  $R$  such that  $d(x, p) \leq R$  for all  $x \in A$ . It is easy to see that once this holds for some  $p \in M$ , it works for all  $p \in M$ , with a choice of  $R$  that depends on  $p$ . The *diameter* of a nonempty bounded subset  $A$  of  $M$  is denoted  $\text{diam } A$  and defined to be the supremum of  $d(x, y)$  over all  $x, y \in A$ .



If  $A, B$  are bounded subsets of  $M$ , then clearly  $A, B$  are  $t$ -close for some positive real numbers  $t$ , and thus the Hausdorff distance  $D(A, B)$  is defined. Of course  $B, A$  are  $t$ -close when  $A, B$  are  $t$ -close, so that  $D(A, B) = D(B, A)$ . Also, if  $A, B, C$  are nonempty subsets of  $M$  and  $s, t$  are positive real numbers such that  $A, B$  are  $s$ -close and  $B, C$  are  $t$ -close, then  $A, C$  are  $s + t$  close, and

$$(5.11) \quad D(A, C) \leq D(A, B) + D(B, C).$$

Suppose that  $\phi(x)$  is a monotone increasing real-valued function on the real line, so that  $\phi(x) \leq \phi(y)$  when  $x, y$  are real numbers such that  $x \leq y$ . For each real number  $x$ , the left and right-sided limits of  $\phi$  at  $x$ , denoted  $\phi(x-)$  and  $\phi(x+)$ , respectively, automatically exist and can be given by

$$(5.12) \quad \phi(x-) = \sup\{\phi(w) : w < x\}, \quad \phi(x+) = \inf\{\phi(y) : y > x\}.$$

Clearly

$$(5.13) \quad \phi(x-) \leq \phi(x) \leq \phi(x+),$$

and  $\phi$  is continuous at  $x$  if and only if

$$(5.14) \quad \phi(x-) = \phi(x+).$$

For each positive real number  $p$ , one can show that the function  $|x|^p$  is a continuous real-valued function on  $\mathbf{R}$ . Fix a positive integer  $n$ , and for each positive real number  $p$  consider the real-valued function  $\|x\|_p$  on  $\mathbf{R}^n$  defined by

$$(5.15) \quad \|x\|_p = \left( \sum_{j=1}^n |x_j|^p \right)^{1/p},$$

$x = (x_1, \dots, x_n)$ . We can also allow  $p = \infty$  here by setting

$$(5.16) \quad \|x\|_\infty = \max(|x_1|, \dots, |x_n|).$$

Thus  $\|x\|_p$  is a nonnegative real number for each  $x \in \mathbf{R}^n$  and  $0 < p \leq \infty$  which is equal to 0 if and only if  $x = 0$ . We also have that

$$(5.17) \quad \|tx\|_p = |t| \|x\|_p$$

for each real number  $t$ ,  $x \in \mathbf{R}^n$ , and  $0 < p \leq \infty$ . Here  $tx$  denotes the usual scalar multiplication of  $t$  and the vector  $x$ , so that

$$(5.18) \quad tx = (tx_1, \dots, tx_n).$$

Clearly

$$(5.19) \quad \|x\|_\infty \leq \|x\|_p$$

for all  $x \in \mathbf{R}^n$  and  $0 < p < \infty$ . More generally, if  $0 < p \leq q < \infty$ , then

$$(5.20) \quad \|x\|_q \leq \|x\|_p.$$

Indeed,

$$\begin{aligned}
 (5.21) \quad \|x\|_q^q &= \sum_{j=1}^n |x_j|^q \leq \|x\|_\infty^{q-p} \sum_{j=1}^n |x_j|^p \\
 &= \|x\|_\infty^{q-p} \|x\|_p^p \leq \|x\|_p^q.
 \end{aligned}$$

When  $0 < p \leq 1$  we can take  $q = 1$  and  $n = 2$  to obtain that

$$(5.22) \quad (a + b)^p \leq a^p + b^p$$

for all nonnegative real numbers  $a, b$ . For  $p \geq 1$  a natural counterpart of this is the fact that  $t^p$  is a convex function of  $t$  on the set of nonnegative real numbers. In other words, if  $t, u$  are nonnegative real numbers and  $\lambda$  is a real number such that  $0 < \lambda < 1$ , then

$$(5.23) \quad (\lambda t + (1 - \lambda) u)^p \leq \lambda t^p + (1 - \lambda) u^p$$

for every real number  $p \geq 1$ .

It is easy to see that

$$(5.24) \quad \|x\|_p \leq n^{1/p} \|x\|_\infty$$

for every  $x \in \mathbf{R}^n$  and every positive real number  $p$ . In fact, if  $p, q$  are positive real numbers such that  $p \leq q$ , then

$$(5.25) \quad \|x\|_p \leq n^{(1/p)-(1/q)} \|x\|_q$$

for all  $x \in \mathbf{R}^n$ . This can be derived from the convexity of the function  $t^{q/p}$  for  $t \geq 0$ .

For any  $x, y \in \mathbf{R}^n$  and  $1 \leq p \leq \infty$  we have that

$$(5.26) \quad \|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

This is easy to derive directly from the definitions when  $p = 1$  or  $\infty$ . In general, one can use homogeneity to reduce to showing that

$$(5.27) \quad \|\lambda x + (1 - \lambda) y\|_p \leq 1$$

when  $x, y \in \mathbf{R}^n$  satisfy  $\|x\|_p, \|y\|_p \leq 1$  and  $\lambda$  is a real number such that  $0 < \lambda < 1$ , and for  $p \geq 1$  this can be derived from the convexity of  $t^p$ ,  $t \geq 0$ .

As a result, when  $1 \leq p \leq \infty$ , we have that

$$(5.28) \quad d_p(x, y) = \|x - y\|_p$$

defines a metric on  $\mathbf{R}^n$ . When  $0 < p \leq 1$  we can set

$$(5.29) \quad d_p(x, y) = \|x - y\|_p^p,$$

and this also defines a metric on  $\mathbf{R}^n$ . To be more precise, this uses the fact that

$$(5.30) \quad \|v + w\|_p^p \leq \|v\|_p^p + \|w\|_p^p$$

when  $v, w \in \mathbf{R}^n$  and  $0 < p \leq 1$ .

If  $(M, d(x, y))$  and  $(N, \rho(u, v))$  are metric spaces and  $f : M \rightarrow N$  is a mapping from  $M$  into  $N$ , then we say that  $f$  is *Lipschitz* if there is a nonnegative real number  $C$  such that

$$(5.31) \quad \rho(f(x), f(y)) \leq C d(x, y)$$

for all  $x, y \in M$ . We might say that  $f$  is  $C$ -Lipschitz in this case, and notice that 0-Lipschitz mappings are constant. Of course Lipschitz mappings are automatically continuous.

Suppose that  $(M_1, d_1(x, y))$ ,  $(M_2, d_2(u, v))$ , and  $(M_3, d_3(z, w))$  are metric spaces, and that  $f_1 : M_1 \rightarrow M_2$  and  $f_2 : M_2 \rightarrow M_3$  are mappings between them. As usual, the composition  $f_2 \circ f_1$  is the mapping from  $M_1$  to  $M_3$  defined by

$$(5.32) \quad (f_2 \circ f_1)(x) = f_2(f_1(x))$$

for all  $x \in M$ . It is easy to see that if  $f_1$  is a continuous mapping from  $M_1$  to  $M_2$  and  $f_2$  is a continuous mapping from  $M_2$  to  $M_3$ , then the composition  $f_2 \circ f_1$  is a continuous mapping from  $M_1$  to  $M_3$ , and that if  $f_1, f_2$  are Lipschitz, then so is the composition  $f_2 \circ f_1$ .

It is easy to generate examples of real-valued Lipschitz functions on the real line, which can then be composed with some of the basic real-valued Lipschitz functions on a metric space mentioned earlier to produce more Lipschitz functions. In general, if  $f_1, f_2$  are two real-valued Lipschitz functions on a metric space  $M$ , then  $f_1 + f_2$ ,  $\min(f_1, f_2)$ , and  $\max(f_1, f_2)$  are also Lipschitz functions on  $M$ , and a real number times a real-valued Lipschitz function is again a Lipschitz function. For products of Lipschitz functions, or reciprocals of nonzero Lipschitz functions, the situation is more complicated, although there are simple sufficient conditions for the result to be Lipschitz.

Now let us look at continuous curves or paths in a metric space. Namely, let  $(M, d(x, y))$  be a metric space, and let  $I$  be a closed interval in the real line. That is,  $I$  might be of the form

$$(5.33) \quad [a, b] = \{x \in \mathbf{R} : a \leq x \leq b\}$$

for some real numbers  $a, b$  with  $a \leq b$ , in which case  $I$  is a closed and bounded interval, or  $I$  might be an unbounded closed interval, of the form

$$(5.34) \quad [a, \infty) = \{x \in \mathbf{R} : x \geq a\}$$

for some real number  $a$ , or

$$(5.35) \quad (-\infty, b] = \{x \in \mathbf{R} : x \leq b\}$$

for some real number  $b$ , or

$$(5.36) \quad (-\infty, \infty) = \mathbf{R}.$$

A continuous path in  $M$  parameterized by the interval  $I$  is simply a continuous mapping from  $I$  into  $M$ . Sometimes we are particularly interested in

paths which are Lipschitz. This can be interpreted as meaning that the path has bounded speed.

Suppose that  $I = [a, b]$  is a closed and bounded interval in the real line, and that  $p : I \rightarrow M$  is a continuous path on  $I$  in the metric space  $(M, d(x, y))$ . By a partition of  $I$  we mean a finite sequence  $\mathcal{P} = \{t_j\}_{j=0}^k$  of real numbers such that

$$(5.37) \quad a = t_0 < \dots < t_k = b.$$

Associated to this partition  $\mathcal{P}$  we get an approximation to the length of the path  $p$ , defined by

$$(5.38) \quad \Lambda_a^b(p, \mathcal{P}) = \sum_{j=1}^k d(p(t_j), p(t_{j-1})).$$

If this quantity is uniformly bounded over all partitions  $\mathcal{P}$  of  $I$ , then we say that the path  $p$  has finite length, and we define the length of  $p$ , denoted  $\Lambda_a^b(p)$ , to be the supremum of  $\Lambda_a^b(p, \mathcal{P})$  over all partitions  $\mathcal{P}$  of  $I$ . If  $p : I \rightarrow M$  is  $C$ -Lipschitz for some nonnegative real number  $C$ , then  $p$  has finite length, and

$$(5.39) \quad \Lambda_a^b(p) \leq C(b - a).$$

It is sometimes convenient to allow  $a = b$  and  $k = 0$  in the definition of a partition, in which case the path automatically has length 0, and in general a path has length 0 if and only if it is constant.

If  $(M, d(x, y))$  is a metric space,  $p$  is an element of  $M$ , and  $r$  is a positive real number, then the *open ball in  $M$  with center  $p$  and radius  $r$*  is denoted  $B(p, r)$  and defined by

$$(5.40) \quad B(p, r) = \{z \in M : d(p, z) < r\}.$$

Similarly, the *closed ball with center  $p$  and radius  $r$*  is denoted  $\overline{B}(p, r)$  and defined by

$$(5.41) \quad \overline{B}(p, r) = \{z \in M : d(p, z) \leq r\}.$$

We make the convention that a “ball” means an open ball unless otherwise specified.

A subset  $U$  of  $M$  is said to be *open* if for every point  $p \in U$  there is a positive real number  $r$  such that

$$(5.42) \quad B(p, r) \subseteq U.$$

The union of any family of open sets is open, and the intersection of finitely many open sets is open. Note that the empty set  $\emptyset$  and  $M$  itself are automatically open subsets of  $M$ , and one can check that open balls in  $M$  are open subsets of  $M$ .

A sequence  $\{x_j\}_{j=1}^\infty$  of points in  $M$  is said to *converge* to a point  $x$  in  $M$  if for every  $\epsilon > 0$  there is a positive integer  $L$  such that

$$(5.43) \quad d(x_j, x) < \epsilon$$

for all  $j \geq L$ . In this case we write

$$(5.44) \quad \lim_{j \rightarrow \infty} x_j = x,$$

and we call  $x$  the limit of the sequence  $\{x_j\}_{j=1}^{\infty}$ . It is not difficult to see that the limit of a sequence is unique if it exists.

A subset  $F$  of  $M$  is said to be *closed* if every sequence of points in  $F$  which converges to some point in  $M$  has its limit in  $F$ . The empty set and  $M$  itself are automatically closed sets, and one can check that closed balls in  $M$  are closed subsets of  $M$ . The intersection of any family of closed sets is closed, and the union of finitely many closed sets is again closed.

In fact, a subset  $U$  of  $M$  is open if and only if its complement  $M \setminus U$  is closed. Recall that the complement of a subset  $E$  of  $M$  in  $M$  is given by

$$(5.45) \quad M \setminus E = \{x \in M : x \notin E\}.$$

Equivalently, a subset  $F$  of  $M$  is closed if and only if  $M \setminus F$  is an open subset of  $M$ .

Suppose that  $M, N$  are sets and that  $f$  is a mapping from  $M$  to  $N$ . If  $A$  is a subset of  $M$ , then the image of  $A$  under  $f$  is denoted  $f(A)$  and is the subset of  $N$  defined by

$$(5.46) \quad f(A) = \{f(x) : x \in A\}.$$

In particular, the image of  $f$  simply means  $f(M)$ . If  $B$  is a subset of  $N$ , then the inverse image of  $B$  under  $f$  is denoted  $f^{-1}(B)$  and is the subset of  $M$  defined by

$$(5.47) \quad f^{-1}(B) = \{x \in M : f(x) \in B\}.$$

The image of the union of a family of subsets of  $M$  under  $f$  is equal to the union of the images of the individual subsets, and the image of the intersection of a family of subsets of  $M$  under  $f$  is contained in the intersection of the images of the individual subsets. The inverse image of the union of a family of subsets of  $N$  under  $f$  is equal to the union of the inverse images of the individual subsets of  $N$ , and the inverse image of the intersection of a family of subsets of  $N$  is equal to the intersection of the corresponding individual inverse images. If  $B$  is a subset of  $N$ , then

$$(5.48) \quad M \setminus f^{-1}(B) = f^{-1}(N \setminus B).$$

If  $(M, d(x, y))$  and  $(N, \rho(u, v))$  are metric spaces, then a mapping  $f$  from  $M$  to  $N$  is continuous if and only if  $f^{-1}(V)$  is an open subset of  $M$  for every open subset  $V$  of  $N$ . This is also equivalent to saying that  $f^{-1}(E)$  is a closed subset of  $M$  for every closed subset  $E$  of  $N$ . Moreover,  $f$  is continuous if and only if for every sequence  $\{x_j\}_{j=1}^{\infty}$  of points in  $M$  which converges to a point  $x$  in  $M$  we have that the sequence  $\{f(x_j)\}_{j=1}^{\infty}$  converges to  $f(x)$  in  $N$ .

A sequence  $\{x_j\}_{j=1}^{\infty}$  of points in a metric space  $(M, d(y, z))$  is said to be a *Cauchy sequence* if for every  $\epsilon > 0$  there is a positive integer  $L$  such that

$$(5.49) \quad d(x_j, x_k) < \epsilon$$

for all  $j, k \geq L$ . Every convergent sequence is a Cauchy sequence, and a metric space is said to be *complete* if every Cauchy sequence in the space converges to some point in the space. A basic property of Euclidean spaces  $\mathbf{R}^n$  with their standard metrics  $\|x - y\|_2$  is that they are complete.

If  $(M, d(x, y))$  and  $(N, \rho(u, v))$  are metric spaces and  $f$  is a mapping from  $M$  to  $N$ , then  $f$  is said to be *uniformly continuous* if for every  $\epsilon > 0$  there is a  $\delta > 0$  such that

$$(5.50) \quad \rho(f(x), f(y)) < \epsilon$$

for all  $x, y \in M$  such that  $d(x, y) < \delta$ . It is easy to see that Lipschitz mappings are uniformly continuous. Constant mappings are uniformly continuous trivially, and the identity mapping on a metric space is 1-Lipschitz and hence uniformly continuous.

If  $f : M \rightarrow N$  is uniformly continuous and  $\{x_j\}_{j=1}^{\infty}$  is a Cauchy sequence in  $M$ , then  $\{f(x_j)\}_{j=1}^{\infty}$  is a Cauchy sequence in  $N$ . In particular, if  $N$  is complete, then  $\{f(x_j)\}_{j=1}^{\infty}$  converges in  $N$ . Also, if  $f : M \rightarrow N$  is uniformly continuous and  $\{x_j\}_{j=1}^{\infty}, \{y_j\}_{j=1}^{\infty}$  are sequences in  $M$  such that

$$(5.51) \quad \lim_{j \rightarrow \infty} d(x_j, y_j) = 0,$$

then

$$(5.52) \quad \lim_{j \rightarrow \infty} \rho(f(x_j), f(y_j)) = 0$$

too.

In any metric space  $(M, d(x, y))$ , the *closure* of a subset  $E$  is denoted  $\overline{E}$  can be defined as the set of points  $x \in M$  for which there is a sequence  $\{x_j\}_{j=1}^{\infty}$  of points in  $E$  which converges to  $x$ . Thus

$$(5.53) \quad E \subseteq \overline{E}$$

automatically, and one can check that  $\overline{E}$  is a closed subset of  $M$  which is contained in any other closed subset of  $M$  that contains  $E$ . A subset  $E$  of  $M$  is said to be *dense* in  $M$  if  $\overline{E} = M$ .

Suppose that  $(M, d(x, y))$  and  $(N, \rho(u, v))$  are metric spaces, with  $N$  complete,  $E$  is a dense subset of  $M$ , and  $f$  is a uniformly continuous mapping from  $E$  to  $N$ . Under these conditions, one can show that there is a uniformly continuous mapping from  $M$  to  $N$  which agrees with  $f$  on  $E$ . This extension is unique, and for that matter if  $f_1, f_2$  are two continuous mappings from one metric space into another, then the set of points in the domain where  $f_1$  and  $f_2$  agree is a closed set.

If  $\{x_j\}_{j=1}^{\infty}$  is a sequence of points in some set  $A$ , and if  $\{j_k\}_{k=1}^{\infty}$  is a strictly increasing sequence of positive integers, then  $\{x_{j_k}\}_{k=1}^{\infty}$  is called a *subsequence* of the original sequence  $\{x_j\}_{j=1}^{\infty}$ . A subset  $K$  of a metric space  $(M, d(x, y))$  is said to be *compact* if every sequence of points in  $K$  has a subsequence which converges to a point in  $K$ . Notice that if  $Y$  is a nonempty subset of  $M$  such that  $K \subset Y$ , then  $K$  is compact as a subset of  $M$  if and only if  $K$  is compact as a subset of  $Y$ , viewed as a metric space on its own, using the restriction of the metric from  $M$ .

A compact subset of a metric space is always closed, basically because any subsequence of a convergent sequence converges to the same limit. Notice that a Cauchy sequence in a metric space which has a convergent subsequence also

converges to the same limit. As a result, a Cauchy sequence contained in a compact subset of a metric space converges to a point in that subset.

A compact subset  $K$  of a metric space  $(M, d(x, y))$  is bounded. To see this, let  $p$  be a point in  $M$ , and assume for the sake of a contradiction that  $K$  is not bounded, so that for each positive integer  $j$  there is a point  $x_j \in K$  with  $d(p, x_j) \geq j$ . It is easy to see that the sequence  $\{x_j\}_{j=1}^{\infty}$  cannot have a convergent subsequence in this case, contradicting the assumption that  $K$  is compact.

A subset  $E$  of a metric space  $(M, d(x, y))$  is said to be *totally bounded* if for every positive real number  $r$  there is a finite set  $F \subseteq E$  such that

$$(5.54) \quad E \subseteq \bigcup_{x \in F} B(x, r).$$

A compact subset of  $M$  is also totally bounded. Indeed, a subset  $E$  of  $M$  is not totally bounded if and only if there is an  $\epsilon > 0$  and a sequence of points  $\{x_j\}_{j=1}^{\infty}$  in  $E$  such that  $d(x_j, x_k) \geq \epsilon$  for all positive integers  $j, k$  with  $j \neq k$ .

More precisely, a subset  $E$  of a metric space  $(M, d(x, y))$  is totally bounded if and only if every sequence of points in  $E$  has a subsequence which is a Cauchy sequence. This is not too difficult to show. As a result, a subset of a complete metric space is compact if and only if it is closed and totally bounded. In particular, closed and bounded subsets of Euclidean spaces are compact.

Let  $(M, d(x, y))$  and  $(N, \rho(u, v))$  be metric spaces, and let  $f$  be a mapping from  $M$  to  $N$ . We say that  $f$  is *bounded* if the image of  $f$  is a bounded subset of  $N$ . The space of bounded continuous mappings from  $M$  to  $N$  is denoted  $C_b(M, N)$ .

There is a natural metric on  $C_b(M, N)$ , called the *supremum metric*, which is defined by

$$(5.55) \quad \sigma(f_1, f_2) = \sup\{\rho(f_1(x), f_2(x)) : x \in M\}$$

for  $f_1, f_2 \in C_b(M, N)$ . Convergence of sequences in  $C_b(M, N)$  is equivalent to *uniform convergence*, as compared to pointwise convergence of mappings. A basic result, which is not too difficult to show, states that if  $N$  is a complete metric space, then so is  $C_b(M, N)$ .

Let us write  $\text{Lip}(M, N)$  for the space of Lipschitz mappings from  $M$  to  $N$ , and for each positive real number  $k$ , let us write  $\text{Lip}_k(M, N)$  for the space of  $k$ -Lipschitz mappings from  $M$  to  $N$ . If  $M$  is bounded, then  $\text{Lip}(M, N)$  is contained in  $C_b(M, N)$ , and for each  $k > 0$ ,  $\text{Lip}_k(M, N)$  is a closed subset of  $C_b(M, N)$ . If  $M$  is bounded,  $p$  is an element of  $M$ ,  $k$  is a positive real number, and  $B$  is a bounded subset of  $N$ , then the set of  $f$  in  $\text{Lip}_k(M, N)$  such that  $f(p) \in B$  is a bounded subset of  $C_b(M, N)$ .

Suppose that  $(M, d(x, y))$  and  $(N, \rho(u, v))$  are metric spaces, with  $M$  compact. If  $f$  is a continuous mapping from  $M$  to  $N$ , then the image of  $f$  is a compact subset of  $N$ . In particular, every continuous mapping from  $M$  to  $N$  is bounded in this case. If  $M, N$  are both compact, then one can show that  $\text{Lip}_k(M, N)$  is a compact subset of  $C_b(M, N)$  for every positive real number  $k$ .

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